Differential quality game with fuzzy information for assessing financial resources for air quality monitoring in cities

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ABSTRACT

Rapid urbanization intensifies pressure on city air quality, making cost-effective monitoring a governance priority. Existing sensor-placement approaches optimize coverage but often ignore strategic behavior of polluters and budget uncertainty, leading to fragile deployments. We propose a decision model that allocates monitoring funds via a bilinear differential game with fuzzy information between an environmental defender and a polluter. Unlike linear differential games solvable via the Cauchy formula, bilinear dynamics and non-measurable adversary strategies require a novel discrete-approximation method within a positional game scheme. The model captures dynamic financial interactions through membership functions and yields analytical characterizations of the defender's preference set and optimal pure strategies. Computational experiments on realistic scenarios illustrate stable funding regimes and support actionable guidance for urban planners: how much to invest, when, and where to expand monitoring stations to achieve resilient oversight under uncertainty. The framework can be embedded in intelligent decision-support tools for smart-city environmental management.

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1. INTRODUCTION

Urbanization is a trend that has been developing for several decades and is likely to continue in the coming years. Despite the fact that cities occupy approximately 2% of the Earth's territory, they are quite effective in terms of economic development. Smart cities, as the next stage of urban development, are an innovative response to the challenges of urbanization. As more and more people move to cities, there is a need to develop efficient urban infrastructure and services supported by advanced information technology (or information technology (IT) [1]. Such IT offers promising solutions [2], in particular, the use of data analytics, the internet of things (IoT), digital twins, and machine learning (ML) technologies [3]. Such IT allows generating and transmitting information between different sectors of smart cities, which, in particular, helps optimize the use of resources, reduce waste and improve the quality of life of citizens [4]. The environmental component plays a key role in the development of the concept of smart cities [5], [6]. The main goal of smart city development is to create a comfortable and sustainable urban environment, in which modern technologies are used to improve the quality of life of the population and optimize the city's functioning processes. Within the framework of this concept, an important place is occupied by monitoring the environmental safety (hereinafter we use the abbreviation environmental safety (ES) of the urban population, since the quality of air, water, and soil directly affects the health and well-being of city residents. To achieve this goal, among other things, air pollution monitoring stations (APMS) (hereinafter referred to as APMS) networks and other sensors placed at strategically important points in the city are used [7], [8]. These devices collect data on air pollution, noise levels, temperature and other environmental parameters.

Air pollution monitoring in smart cities is a pressing challenge driven by rapid urbanization and rising emissions of pollutants. Optimizing the placement of air pollution control stations (APCS) is a mathematically complex problem because it must account for numerous parameters, including pollution sources, airflow patterns, building density, and budget constraints [9]. Existing sensor-placement methods rely primarily on data interpolation [10]-[12], ML [13], and data-mining techniques [14], [15]. However, most solutions do not consider the strategic interaction among stakeholders, which leads to suboptimal resource allocation. This study proposes an approach based on differential quality games with fuzzy information that captures the conflict of interests between the parties and optimizes the monitoring strategy under data uncertainty. The solution to the problems of ensuring effective air quality monitoring (AQM) in smart cities is impossible without funding. Funding of means and activities to maintain AQM, as well as accounting for financial damage from possible natural and anthropogenic factors are interdependent procedures that determine the level of the environment and the quality of life of society. The party seeking to improve the environmental situation needs financial resources (FR) to purchase and upgrade AQM tools, pay for personnel services, implement advanced technologies (such as the IoT), data science, and ML), as well as to upgrade APMS and their optimal location. The party that negatively influences the air condition is characterized by FRs that determine the possible damage from various factors, often of an unforeseen nature. This party may include not only natural phenomena, but also enterprises and groups of people who, striving for maximum profit, neglect environmental protection technologies, including air, or damage the ecology without taking into account the consequences. These circumstances briefly characterize the “conflict” interaction of the parties. To find rational behavior of the parties, primarily the party engaged in AQM in cities, game theory serves as an effective tool for analysis and recommendations. This paper considers a multidimensional differential game with fuzzy information, where two parties (two players) participate. The first party (a player, for example, with the conventional name “environmental defender”) is a player who distributes FR for the location of APMS. The goal of the first party is to effectively distribute funds to create an APMS network that minimizes air pollution and maximizes the environmental situation in the city. The second party (a player, for example, with the conventional name “polluter”) is nature and anthropogenic factors. This party manages finances that characterize the amount of damage from various states of nature, such as cataclysms and anthropogenic activity (for example, emissions from factories, and transport). The state of the second player is fuzzy, which means that it belongs to a fuzzy set and is characterized by uncertainty.

2. LITERATURE REVIEW

Many years of previous research provide valuable experience and practical recommendations for using mathematical methods and algorithms to optimize the placement of sensors for AQM, including in smart cities, and studying these approaches will help to understand which methods have proven most effective and why, which will help to avoid repeating mistakes and implement best practices.

Thus, in the work [9] the authors focus on the factors influencing air flows and the spread of pollutants in urban development, as well as the use of simulations and modeling to assess these factors. The study focuses on the physical and technical aspects of the problem. The work does not consider the application of game theory. Works [10]-[12] show how optimization and ML methods can help to determine the best locations for weather stations and other sensors to cover the city area as efficiently as possible and obtain relevant data. Data analysis helps to understand which areas of the city are most vulnerable to pollution and require special attention. In particular, Hassani *et al.* [10] compares different methods, including interpolation, satellite-based ML, and meteorological modeling, for mapping air temperature in a smart city. The study uses data from the Netatmo network of IoT weather stations in Warsaw, Poland, and evaluates the effectiveness of these methods in predicting fine-scale temperature variations, showing the advantages and limitations of each approach. Du *et al*. [11] provides a detailed review of the deployment and management of AQM systems in smart cities. The paper discusses the key aspects and challenges associated with the creation and operation of sensor networks for smart cities. The main topics covered in the paper include: sensor network technologies (discussion of different types of sensors used to monitor the environment, infrastructure, and various processes in cities); sensor deployment methods (an overview of strategies and algorithms used to optimally deploy sensors to ensure maximum coverage and data accuracy); data management and analysis (methods for collecting, storing, and processing data obtained from sensor networks, including the use of ML and data analytics to obtain useful information). The study highlights the importance of integrating various technologies and methods to create effective and sustainable monitoring systems in smart cities.

Bacco *et al*. [12], the issue of using technologies for environmental monitoring in smart cities is raised. The paper focuses on environmental monitoring. The use of various sensors and networks for environmental monitoring is considered. The publication also touches upon IoT technologies. IoT technologies used to collect and transmit environmental data are described, and methods for optimizing the placement of sensors for efficient coverage of the smart city territory and obtaining relevant data are analyzed. The paper provides examples of successful implementation of monitoring systems in various cities, including analysis of data obtained using these systems. Zhu and Zhao [13], aspects of how data science technologies allow integrating data from various sources, including weather stations, satellites, and transport systems, to create a comprehensive picture of the city's environmental condition are considered. The authors note that data visualization, including geographic information systems (GIS), helps citizens and authorities better understand the environmental situation and make informed decisions aimed, among other things, at protecting the environment. Based on data analysis and ML forecasts, city services can implement smart solutions for traffic management, reducing emissions from vehicles and industry, and improving green spaces.

The closest works to our research are [14]-[16]. Thus, Sheng *et al*. [14] consider the problem of coordinating the interests of various stakeholders in environmental regulation using evolutionary game theory. The work focuses on the challenges and strategies in coordinating the goals and actions of various stakeholders, such as government agencies, enterprises and the public, to achieve effective environmental policies, and although this work is based on game theory, it does not specifically concern the application of game models to finding optimal resource allocation strategies for the placement of AQM stations in smart cities, focusing on broader problems of coordination and regulation of environmental management. Eryganov *et al*. [15] consider models of cooperation between waste producers under conditions of limited or prohibited use of landfills. The study uses the apparatus of cooperative game theory to formalize the decision-making process aimed at reducing the costs of processing non-recyclable solid waste. The work describes various classes of cooperative games that model the interaction between waste producers with different restrictions on cooperation. Fan and Hui [16] the mechanisms of decision-making by governments and developers in the area of green building incentives are considered. The authors used evolutionary game theory to model the evolutionary behavior of two players, which allows quantitatively illustrating the effectiveness of incentives and changes in players' strategies.

It should be noted that the problem of ensuring ES, including the task of monitoring air quality, at a high level leads to a continuous search for tools to support it. Some of the effective tools are intelligent information systems (IIS), which are based on mathematical approaches, such as optimization methods, game theory methods, and others. In our opinion, approaches to solving ES problems based on the use of the apparatus of differential quality games taking into account fuzzy information are those that will make a significant contribution in this direction. That is why this paper considers the solution to the problem of ensuring ES within the framework of the differential quality game scheme with fuzzy information.

3. THE PURPOSE AND OBJECTIVES OF THE STUDY

Purpose–development of a mathematical model for assessing FR for monitoring air quality in cities based on approaches to solving differential quality games with fuzzy information. The article addresses the issues aimed at:

- Development, based on the differential quality game, of a model for finding optimal financial strategies of players in the process of monitoring the state of urban air.
- Conducting computational experiments to test the model's performance.

4. METHOD

4.1. Problem statement

The solution to the problems of ensuring effective AQM in smart cities, as a component of ES policy, is impossible without proper funding. Financing of funds and activities to support ES, as well as accounting for financial damage from possible natural and climatic factors, negative consequences of anthropogenic factors, is an interdependent procedure. It is this procedure that has a decisive influence on the level of environmental pollution and the quality of life of society. The party that seeks to improve the environmental situation needs financial resources (hereinafter FR) to purchase and/or upgrade the tools for maintaining ES, to pay for the services of personnel of the units that provide ES, to implement advanced technologies, including for the development of the APMS network and optimization of their location. For the party that has a negative impact on ES, FR are described that determine the possible damage from certain factors, often of an unforeseen nature. This party includes not only natural phenomena, but also a part of society that, in pursuit of maximum profit, neglects the implementation of environmental protection technologies, or damages the environment without thinking about the consequences. An example is an enterprise located in the city, polluting the atmosphere with emissions without proper purification, in an effort to reduce production costs. As a result, the surrounding air becomes heavily polluted, which leads to deterioration in public health and increased costs for medical care for city residents. In addition, this enterprise does not take into account potential fines and sanctions from environmental services, which may lead to significant financial losses in the future. The circumstances outlined briefly characterize the "conflict" interaction of the parties. To find rational behavior of the parties, primarily for the party protecting the ES, game theory serves as an effective tool for analysis and recommendations.

In this work, the opposing parties, according to the postulates of game theory, will be called players. The first player is the environmental defender (hereinafter we will use the term "environmental defender"). The second player is the party that causes damage to the environment. We will call him "the polluter." The paper will only consider the financial aspect of the confrontation, without touching on the technological and technical aspects used by both parties.

The interaction of players occurs continuously in time, let us describe it. The environmental defender has M technical and technological strategies to counteract the polluter (the second player). For example, these may be strategies related to: optimal placement of APMS and the creation of a network of sensors and stations for real-time AQM, as in the air quality egg project [17], using IoT technologies, in particular the implementation of IoT sensors to collect and transmit air pollution data to a centralized analysis system, as is done in Barcelona [18], the use of data science and ML techniques, such as analyzing collected data using ML to predict pollution levels and take proactive measures, as in the London air quality network project [19], the creation and increase of green spaces, such as supporting urban parks and green roofs to reduce air pollution, as in Singapore [20], innovative air purification technologies and the use of filtering systems and devices, such as the street air purification systems implemented in Beijing [21], campaigns to raise public awareness of air pollution problems and involve citizens in monitoring and improving the environmental situation, as in Houston and others [22].

The polluter (the second player), in order to cause damage to the environmental defender, also has M technical and technological strategies. For example, industrial companies may hide real emissions data or manipulate sensor readings to avoid fines, as happened to car manufacturers in the Diesel gate scandal [23].

Companies may also use outdated technology and equipment, which leads to increased emissions of pollutants, as is the case in many plants in developing countries [24]. This may include failure to install or improper use of emission control systems to reduce costs, as has been found in some chemical plants in China [25]. Lobbying and political pressure strategies are also common, such as pressuring government bodies to weaken environmental regulations and standards. A situation with hidden emissions and/or illegal discharges is possible. For example, illegal emissions of pollutants at night or in remote areas have been recorded at a number of plants in India [26]. A criminal strategy of providing dishonest reports and falsifying data is also possible. For example, when companies provide false reports on the amount and composition of emissions [27]. We can also mention the use of cheap and toxic materials in production, which leads to significant emissions of harmful substances, as happens in textile factories in Bangladesh [28]. The above and other strategies allow the polluter to reduce its costs and maximize profits at the expense of ES and the health of the population of smart cities. The assumption of equality of strategies is not essential, since the inequality of strategies is easily reduced to their equality. The interaction process occurs as follows. Players begin to apply their technological and technical strategies. Their application affects the financial state of the players, namely, the implementation of the second player's technological and technical strategies will lead to financial losses for the environmental defender. The implementation of the environmental defender's technological and technical strategies will lead to his financial income. Let us specify this circumstance.

The implementation of the *i*-th technological or technical strategy by the polluter brings financial damage to the environmental defender in the amount of ϕ_i^1. The implementation of the *i*-th technological or technical strategy by the environmental defender brings financial income in the amount of ϕ_j^2. Let denote by r_{ij}^1 the relation ϕ_i^1/ϕ_j^2, and by the r_{ij}^2 the relation θ_j^2/θ_i^1. If $\theta_i^1 = 0$ for some i or $\phi_j^2 = 0$ for some j, then such strategies are excluded from consideration. Let introduce a matrix Δ_1, consisting of elements r_{ij}^1. The number of rows corresponds to the number of technological and technical strategies of the polluter. Here the number of each row corresponds to the corresponding technological or technical strategy of the polluter. The matrix Δ_2 consists of elements that form rows that correspond to the technological or technical strategies of the environmental defender. Then the columns correspond to the technological or technical strategies of the polluter. That is, the elements r_{ij}^2 mean that they are in *j*-th row and in *i*-th column.

Matrices Δ_1 and Δ_2 are analogs of the so-called profitability matrices. In the matrix Δ_1 each element characterizes the amount of losses from the application of the attacking player's technological or technical strategy, which is compensated by an income set from the application of the defending player's technological or technical strategy. The same is true for the elements of the matrix Δ_2. To form the dynamics of interaction, we introduce the following notations:

By μ_j $(j = 1, \dots, M)$ such values as$\mu_j \geq 0, \sum_{j=1}^{M} \mu_j = 1$, which are elements of a M -order diagonal matrix Ξ, with diagonal elements μ_j . The matrix Ξ characterizes the "structure" of the polluter's FR set. The element μ_j denotes the share of the j -th value of the environmental defender's FR set, which shows the transformation of this set into the j -th component of the value of the polluter's FR set. That is, if $(z_1, \dots, z_M)$ is the environmental defender's FR set, then the environmental defender's income set, which is equal to $\mu_j \cdot (z_1, \dots, z_M)$, will be transformed into the j -th component of the value of the polluter's FR set;

By ρ_j $(j = 1, \dots, M)$ such values as $\rho_j \geq 0, \sum_{j=1}^{M} \rho_j = 1$, which form a M-order diagonal matrix Λ, with diagonal elements λ_j . The matrix Λ characterizes the "structure" of the environmental defender's FR set. The element ρ_j denotes the share of the j-th value of the environmental defender's FR set, which shows the transformation of this set into the j-th component of the value of the environmental defender's FR set. Consequently, if $(w_1, \dots, w_M)$ is the polluter's FR set, then the polluter's income set, which is equal to $\rho_j \cdot (w_1, \dots, w_M)$, will be transformed into the j-th component of the value of the environmental defender's FR set. Let us formulate the following remark.

a. Remark

If there is the environmental defender's FR set $z = (z_1, \dots, z_M)$, then if we perform the operation: $\Delta_1 \cdot z$, we will get a M -dimensional vector. This vector seems to denote the polluter's FR set. However, in reality, this product allows determining only one component of this M -dimensional vector of the polluter. This is a consequence of the fact that the entire vector $z = (z_1, \dots, z_M)$ will be "spent" on this component alone. For other components of the polluter's FR set there is no more environmental defender's FR set "equivalent" to this component of the polluter. The entire environmental defender's FR set "went" to "equilibration" in efficiency with one component of the polluter's FR set. Thus, it is necessary to break the environmental defender's FR set into parts. This will allow "equilibration" of the polluter's FR set for all its components. This is done by introducing the following set: μ_j $(j = 1, \dots, M)$: $\mu_j \geq 0, \sum_{j=1}^{M} \mu_j = 1$. Using other methods, it is also possible to select these coefficients.

Similar reasoning is also valid for the urban air polluter's FR set. The meaning of the introduced matrices will become clear in the process of describing the dynamics of interaction. We will describe it below. Thus, the environmental defender, having FR at the moment of time $t \in [0, +\infty)$ $z(t) \in R_+^M$, ransforms them to the value of resources $L_1 \cdot z(t)$. Each component of the FR vector means the value of resources spent on the implementation of the corresponding technological or technical strategy of this player. Here $L_1 -$ the M -order transformation matrix of the environmental defender's FR set, with positive elements (analogous to the FR growth rate). Next, he determines the value of his investment $U(0) \cdot L_1 \cdot z(t)$ by selecting the elements $u_i(t)$: $0 \leq u_i(t) \leq 1$, that are the diagonal elements of the M -order diagonal matrix. They are the ones that determine the values of the environmental defender's investment strategy. This means that this is not a technological or technical strategy of an environmental defender, but an investment. Such environmental defender's investment means that it allows compensating for the $\Xi \cdot \Delta_1 \cdot U(0) \cdot L_1 \cdot z(t)$ losses from the polluter's actions. This is expressed in the fact that the dynamics of the change in the polluter's FR leads to a decrease in its FR and, consequently, to compensation for the environmental defender's financial losses from the polluter's actions.

It is assumed that in the interaction of players, from an informational point of view, there is a situation in which the environmental defender does not know the exact state $w^{\xi}(0)$ $(w^{\xi}(0) \in int\, R_+^M)$ of the polluter at a given moment in time $t = 0$. He only has access to information that the state of the polluter belongs to a fuzzy set $\{\Omega, m(.)\}$, where Ω is the subset $R_+^M, m(.)$ −the membership function of the state w^{ξ} to the set $\Omega, m(w^{\xi}) \in [0,1]$ for $w^{\xi} \in \Omega$. Note that the fuzziness of the set can be partially justified by the fact that natural and climatic factors can affect the availability of complete information. For an intuitive understanding of fuzzy sets, consider the following example. Suppose the air-pollution level is evaluated on a scale from 0 to 100. In a classical (crisp) approach, a value of 55 can be assigned strictly to the category "moderate pollution." In fuzzy logic, the same value can simultaneously belong to the sets "low pollution" with a membership degree of 0.4 and "moderate pollution" with a membership degree of 0.6. This reflects uncertainty in how the boundaries between categories are perceived. The applied formalization of information fuzziness follows a standard approach, wherein a set to which the variable belongs is specified, along with a corresponding membership function that characterizes the degree of certainty with which the variable is associated with this set.

The polluter acts in a similar way. Having FR at the moment of time $t \in [0, +\infty)$, $w^{\xi}(t) \in R_{+}^{M}$ he transforms them to the value of resources $L_2 \cdot w^{\xi}(t)$. Each component of the resource vector means the value of FR, which goes to the implementation of the corresponding technological or technical strategy of this player. Here $L_2 - M$-order matrix of transformation of the polluter's FR, with positive elements. Then he determines the value of his investments $V(t) \cdot L_2 \cdot w^{\xi}(t)$. This is possible by selecting elements $v_i(0): 0 \leq v_i(0) \leq 1$, that are diagonal elements of the M -order diagonal matrix $V(0)$. The latter determine the values of the polluter's investment strategy (note that this does not mean the polluter's technological or technical strategy). Such an investment of the polluter allows leveling out the $\Lambda \cdot \Delta_2 \cdot V(t) \cdot L_2 \cdot w^{\xi}(t)$ income from the actions of the environmental defender. This is expressed in the fact that the dynamics of the change in the FR of the environmental defender leads to a decrease in his FR and, consequently, to compensation for the polluter's losses from the environmental defender's tools.
Then the players FRs at the moment of time $t \in [0, +\infty)$ satisfy the following system of differential:

$$dz(t)/dt = -z(t) + L_1 \cdot z(t) - U(t) \cdot L_1 \cdot z(t) - \Lambda \cdot \Delta_2 \cdot V(t) \cdot L_2 \cdot w^{\xi}(t);$$
$$dw^{\xi}(t)/dt = -w^{\xi}(t) + L_2 \cdot w^{\xi}(t) - V(t) \cdot L_2 \cdot w^{\xi}(t) - \Xi \cdot \Delta_1 \cdot U(t) \cdot L_1 \cdot z(t) \quad (1)$$

At the point in time t ($t \in [0, +\infty)$) the following options are possible:

$(z(t), w^{\xi}(t)) \in S_0$, with reliability of $\geq p_0$, $(0 \leq p_0 \leq 1)$ (2)

$(z(t), w^{\xi}(t)) \in F_0$, with reliability of $\geq p_0$ (3)

$\left(z(t),\ w^{\xi}(t)\right) \in D_0$ with reliability of $\geq p_0$ (4)

$\left(z(t),\ w^{\xi}(t)\right) \in H_0$ with reliability of $\geq p_0$ (5)

where

$$S_0 = \cup_{i=1}^{2\cdot M}\{(z,w): (z,w) \in R^{2\cdot M}, z \succ 0, w_i = 0\}$$

$$F_0 = \cup_{i=1}^{2\cdot M}\{(z,w): (z,w) \in R^{2\cdot M}, z = 0, w_i \succ 0\}$$

$$D_0 = \{\cup_{i=1}^{2\cdot M}\{(z,w): (z,w) \in R^{2\cdot M}, z_i = 0\}\} \cap \{\cup_{i=1}^{2\cdot M}\{(z,w): (z,w) \in R^{2\cdot M}, w_i = 0\}\}$$

$$H_0 = int\, R_{+}^{2\cdot M}$$

If (2) is met, then we consider that the air pollution monitoring procedure is completed, since the polluter did not have enough FR to cause damage to the environmental defender with at least one of his technological or technical strategies that the second player planned to use, with reliability of $\geq p_0$. If (3) is met, then we consider that the air pollution monitoring procedure is completed, since the environmental defender did not have enough FR to counteract the damage caused by the polluter. This is true, at least for one of his technological or technical strategies, which the environmental defender planned to apply, with reliability of $\geq p_0$. In (4) means that the players did not have enough FR with reliability of $\geq p_0$ to continue the interaction at least on one of their technological or technical strategies. The interaction is over. If (2)-(4) are not met, then the interaction of the players continues.

The process described by system (1) for the air pollution monitoring financing procedure is considered within the framework of the positional differential game scheme with fuzzy information [29]. Due to symmetry, we will limit ourselves to considering the problem from the position of the environmental defender. The second problem is solved similarly. The definition of a pure strategy and the set of environmental defender's preference was given in [29]-[31].

The solution to problem 1 consists of finding the sets of environmental defender's "preference" V_1^* and his optimal strategies $U_*(.)$. The problem is posed similarly from the polluter's position. The environmental defender in task 1 is considered an ally player, the polluter is considered an enemy player. In task 2–vice versa. The task of the procedure for interaction of players by means of a system of differential equations generates at each moment in time t a set of pairs of fuzzy sets $\{\Gamma_t, n_t(.)\} \times \{\Omega_t, m_t(.)\}$ that reflect the process of transition from the initial states of the players $(x(0), y^{\xi}(0))$ to the subsequent ones, when the players apply control actions.

It is assumed that the environmental defender knows his states $z(\tau)$ for $\tau \leq t$ at each moment $t\,(t \in [0, +\infty))$. The following conditions are satisfied: $z(\tau) \succ 0$, if the reliability of such states are $n_\tau(z(\tau)) \geq p_0$ and $z(\tau) \notin int\, R_+^M$, and if the reliability of such states $n_\tau(z(\tau)) \prec p_0$, and the values of the realizations of the environmental defender's strategy $U(\tau)(\tau \leq t)$ allocated for interaction with the second player are known. Let us define the function $F(.): Z \rightarrow R_+, F(z) = \{sup\, m\,(y), для\, y \leq z, z \in R_+^M\}$. Let us denote by Φ–the set of such functions, by $T^* = [0, +\infty)$–the time interval.

b. Definition

A pure strategy $U(.,.,.)$ of the environmental defender is a set of functions $u_i(.,.,.): T^* \times R_+^M \times \Phi \rightarrow [0,1], (i = 1, \ldots, M)$, such that $u_i(t, z, F) \in [0,1], (t \in T^*,\ z \in R_+^M,\ F \in \Phi)$. In other words, a pure strategy of an environmental defender is a specific set of actions that this player performs to achieve his goals of improving the environmental situation. In this case, the environmental defender uses his FR and technological capabilities to effectively place and modernize APMS. Thus, the pure strategy of the environmental defender assumes consistent and systematic implementation of actions in order to minimize the level of air pollution and improve ES in smart cities. The polluter (the second player) chooses his strategy $V(.)$ based on any information. The environmental defender seeks to find a set of his initial states that have the following property.

Property: if the game starts from such initial states, then the environmental defender can, by choosing his strategy $U_*(.)$, ensure that (2) is met at one of the moments in time t. In this case, this strategy, chosen by player 1, helps to prevent the second player from meeting condition (3) at previous moments in time.

The set of such states will be called the set of environmental defender's preference V_1^*, and the environmental defender's strategies $U_*(.)$ that possess the specified properties will be called his optimal pure strategies. The goal of the environmental defender is to find the set of preference, as well as to find his strategies, by applying which he will achieve the fulfillment of (2).

The formulated game model according to the classification of decision theory corresponds to the problem of decision making under fuzzy information conditions. Such a model is a bilinear differential game of quality with several terminal surfaces. Finding the sets of environmental defender's preference and his optimal strategies depends on many parameters.

It should be noted that, to solve the proposed bilinear differential game, we developed a method based on a limiting procedure applied to solutions of multistage games, which made it possible to overcome the constraints encountered when attempting to address the problem with existing approaches. Because the system of equations is bilinear, methods for linear games that rely on the Cauchy formula are inapplicable here. Moreover, since the problem may involve non-measurable control strategies, the standard apparatus of positional differential games cannot be used directly, despite the existence of a value in the "small" (instantaneous) game. These circumstances justify the claim that the work is not only original in its problem formulation but also contributes to the methodology for solving bilinear differential quality games. Further novelty is provided by the information structure of the problem—namely, its fuzziness [29], which brings the model closer to real-world conditions.

To describe the sets of environmental defender's preferences, it is necessary to introduce a number of notations and quantities. Let us define the set $C(p_0) = \{c(0): F(c(0)) \geq p_0\}$. For any $z \in R_+^M$ let's consider the set $L_z = \{\gamma: \gamma = l \times z, l \in R_+\}$. For any $z \in R^M$ let's consider the set $Q(z, p_0) = C(p_0) \cap L_x$; Let's define a vector $\delta(z, p_0): \delta(z, p_0) = inf\{\delta^*: \delta^* \in Q(z, p_0)\}$. Let's consider the set $\Delta(p_0) = \{\delta(p_0): \exists z \in R_+^n: \delta(p_0) = \delta(z, p_0)\}$. Let us present the conditions that make it possible to find a solution to the game, i.e., the sets of "preference" V_1^* and the optimal strategies $U_*(.)$ of the environmental defender. The aforementioned sets and vectors serve, to a certain extent, as a rule base for handling the parameters that define the fuzziness of information, thereby enabling the analytical representation of the solution to the problem under consideration.

4.2. The methodology used to solve the problem

This study applies a differential quality game with fuzzy information to solve the problem of assessing AQM resources in smart cities. The use of differential equations allowed describing the dynamics of changes in the system state over time, i.e., modeling the monitoring efficiency depending on the actions of both players. This approach takes into account time aspects and adaptive strategies of the parties, which is important for long-term planning and management of ES of cities.

Thus, the proposed methodology provides a systematic and integrated approach to solving the problem of assessing and managing AQM resources, which helps to increase the efficiency of ES and improve the quality of life in smart cities.

4.3. Restrictions adopted when solving the problem

In this formulation of the problem for improving ES in smart cities, restrictions are introduced due to the conceptual nature of the second party – "Nature". In game theory, traditionally, FRs cannot be directly applied to nature. Therefore, the “financial resources” of the second party are interpreted metaphorically: nature “pays” for the deterioration of environmental conditions and the health of the ecosystem through the following factors: pollution effects (acid rain, water pollution, and reduction of biological diversity); negative consequences for public health (deterioration of air quality leads to an increase in illness among people, which can be considered as a “cost” from nature, affecting public health). These factors cannot be predicted with absolute accuracy and are described with some degree of uncertainty. Thus, the restrictions are related to the fact that the second player (polluter) is an aggregate player in the form of enterprises and/or their management, i.e. a group of people spoiling the environment and operating FR taking into account natural conditions, presented as a fuzzy set.

4.4. Solution to the task 1

The solution to the problem depends on the ratio of parameters that determine the procedure for financial confrontation between the ally player and the second opponent player in the air pollution monitoring problem. In this paper, we will provide an analytical solution to the problem for one of the variants of the game parameters ratio. For other variants, the solution will be found similarly.

Let us denote by S_1 the matrix $\Xi \cdot \Delta_1$, by S_2 the matrix $\Lambda \cdot \Delta_2$.

Let the following game parameters ratio be satisfied:

$$L_2 \cdot S_1 \geq S_1 \cdot L_1, S_1 \cdot L_1 \cdot S_2 \geq L_2, \sum_{j=1}^{M}(S_1 \cdot L_1)_{ij} \geq \sum_{j=1}^{M}(L_2)_{ij},\ 1 \leq i \leq M; L_1 \succ 0,\ L_2 \succ 0;$$

(matrix inequalities are considered in the ratio),

$$\sum_{\theta=1}^{M}[(S_2 \cdot L_2)_{i\theta}/[\sum_{j=1}^{M}(S_2 \cdot L_2)_{ij}] \times [\sum_{j=1}^{M}[(S_1)_{\theta j}] \prec \sum_{j=1}^{n}[(S_2)_{ij}],\ 1 \leq i \leq M \quad (6)$$

Let us denote by $(\gamma_{sred})_i$ the left side of the inequality (6), by $(\delta_{sred})_i$ - the right side of the inequality (6), by q_i^* the value $\sqrt{(\gamma_{sred})_i/(\delta_{sred})_i}$.

Then the set of environmental defender's preferences V_1^* will be determined as:

$$V_1^* = \cup_{i=1}^{M}\{V_1^i \cap V_1\};$$

where

$$V_1^i = \{(z(0), \delta(p_0)) : (z(0), \delta(p_0) \in R_+^{2M},\ [\sum_{j=1}^{M}(L_2)_{ij}] \times \delta_i(p_0) \prec [\sum_{j=1}^{M}(S_1 \cdot L_1)_{ij} \cdot z_j(0)];\ 1 \leq i \leq M;$$

$$V_1 = \cap_{i=1}^{M}\{(z(0), \delta(p_0)):\ q_i^* \times z_i(0) \geq [\sum_{j=1}^{M}(S_2 \cdot L_2)_{ij} / [\sum_{j=1}^{M}(S_2 \cdot L_2)_{ij}] \times \delta_j(p_0)\};$$

The optimal strategy of an environmental defender $U_*(z, \delta) = E$ is in the area V_1^* and is not defined outside this area.

The solution to the task 1 is found in a completely similar way for other game parameters ratio. The solution to the task 2, from the polluter's point of view, is found in the same way.

Thus, a solution to the bilinear differential game for the case of multidimensional variables and fuzzy information in an explicit form, which is a very difficult task, was found. This made it possible to solve the problem of finding strategies in the task of monitoring air pollution, in the case of fuzzy information caused by the need to ensure ES of objects. In addition, the obtained result makes it possible to predict the result of air pollution monitoring in practice, when there is no crisp information on the financial state of the polluter. The derived theoretical solution enables drawing conclusions regarding the sensitivity of the problem to parameter variations. It is evident that when the players' states lie on the boundaries of the preference sets, any modification of these parameters may lead to a fundamentally opposite outcome in the players' interaction.

The incorporation of fuzzy information fundamentally affects the equilibrium strategies. Unlike crisp models where the defender knows the exact state $y(t)$, here the membership function $\mu\_M(y)$ introduces a confidence level for each possible state. This leads to a preference set W_1 that accounts for the worst-case scenario within the fuzzy region, making strategies more conservative but robust to incomplete information. The bilinear structure $A(u)x + B(v)y$ prevents the use of classical Cauchy-formula-based solutions and requires specialized discrete-approximation methods, which constitute the methodological novelty of this work.

5. COMPUTATIONAL EXPERIMENT

It should be noted that the model presented above is, at the current stage of research, predominantly of a game-theoretic nature. Unlike empirical studies focused on the collection of field data, the proposed formulation is grounded in a rigorous mathematical framework of differential quality games with fuzzy information. Its applicability is demonstrated through computational experiments, which serve as a functional analogue of empirical validation for models of this type. This approach aligns with established practices in game theory [30]-[33], where the correctness and utility of a model are evaluated not through direct observation, but via scenario reproducibility, robustness of strategies, and interpretability of the obtained results in the context of environmental management tasks. At the same time, the use of approximation procedures and simulations on test data (see Table 1) provides evidence of the practical relevance of the proposed approach. This, in turn, lays the foundation for its further integration into real-world monitoring systems and decision support tools.

Table 1. Results of computational experiments

Experiment no	Input data	Result
1	z_1=1, z_2=2, δ=4, l_1=2, l_2=0.5, s_1=1, s_2=5	The first player did not reach the goal
2	z_1=1, z_2=1, δ=1, l_1=2, l_2=5, s_1=2, s_2=6	The first player did not reach the goal
3	z_1=2, z_2=1, δ=6, l_1=1, l_2=0.5, s_1=3, s_2=7	The first player did not reach the goal
4	z_1=3, z_2=5, δ=3, l_1=5, l_2=0.5, s_1=8, s_2=1	The first player has reached the goal

To test the efficiency of the proposed model of the differential quality game with fuzzy information, a computational experiment was conducted, see Table 1. The main goal of the experiment was to visualize the trajectories of the polluter depending on the strategies and resources of both parties of the game. During the experiment, an assessment of the impact of various strategies of the environmental defender on the environmental situation (air pollution) in the city was also made. The model is implemented in the Python programming language, using the NumPy, SciPy, Matplotlib, and other libraries.

For the cybernetic implementation of the model, triangular-type membership functions were employed. This choice ensured computational simplicity of approximation and interpretability of the results. Each linguistic term representing the pollution level (low, moderate, and high) was described by a system of overlapping functions within a normalized range of [0,1]. This allowed for class overlap to be accounted for and enabled defuzzification using the center-of-gravity method. The adopted approach provided a balance between computational efficiency and approximation accuracy, making the model applicable to monitoring tasks with limited computational resources.

Numerical experiments demonstrated that the equilibrium strategies of the players remain robust under ±10% variations in the parameters of the dynamic matrices. Minor fluctuations in the coefficients affect only the convergence rate, without altering the qualitative nature of the players' preferences or the structure of the set of attainable outcomes. An exception occurs in boundary-case scenarios, where the initial state of resources approaches critical thresholds—under such conditions, the equilibrium tends to shift toward more aggressive strategies by the opposing side.

The employed discrete-approximation method exhibits polynomial complexity with respect to the number of approximation steps and the dimensionality of the strategy space. While computational costs increase with the number of players or the expansion of the control set, they remain manageable for problems of moderate scale. For systems involving a large number of agents, method adaptation would be required. In the long term, this could be addressed by leveraging parallel computation or applying dimensionality reduction techniques.

The graphical results in Figure 1, represented as hyperplanes, enabled us to determine the eco defender’s preference set in the game’s parameter space. The hyperplane boundaries delineate regions of admissible states where equilibrium is attained in the bilinear differential quality game. Their intersection describes the set of feasible eco defender strategies that secure a minimax payoff under uncertainty. These results made it possible to verify the controllability conditions of the monitoring system and to identify zones of optimal FR allocation within the dynamic funding system for AQM in cities of Ukraine and Kazakhstan.

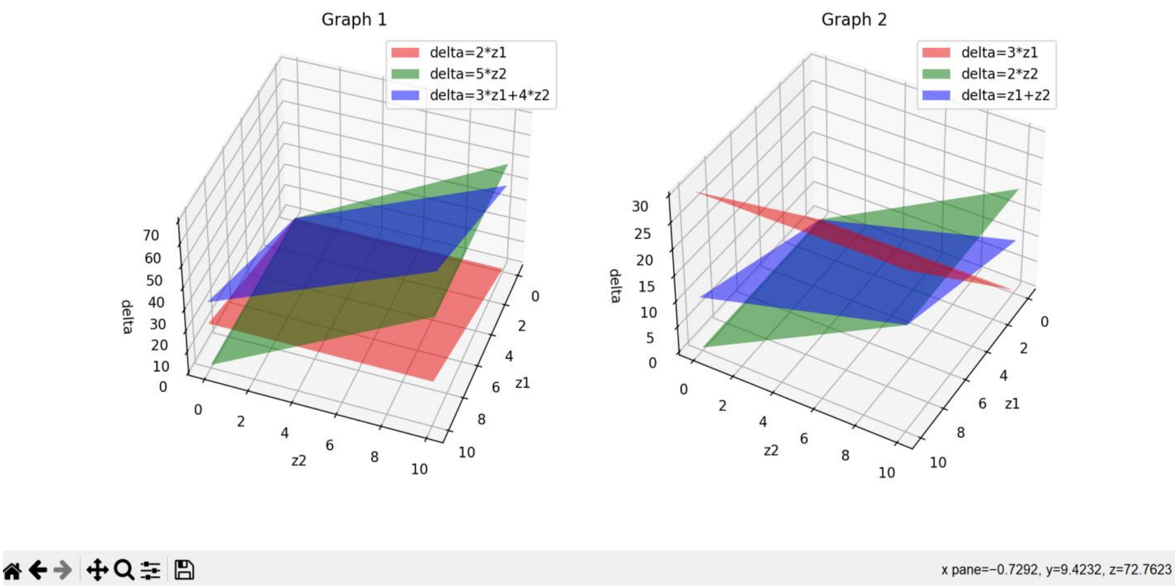

Figure 1. Results of the computational experiment

6. DISCUSSION OF THE RESULTS OF THE COMPUTATIONAL EXPERIMENT

Figure 1 shows the preference sets of the first player (the environmental defender). These sets consist of states in the positive three-dimensional space bounded by three hyperplanes. The hyperplanes define the boundaries within which the environmental defender can achieve their goals. Pairwise intersections of these hyperplanes form the so-called equilibrium (balance) rays that describe the optimal interaction between the environmental defender and the polluter. When the players' states lie on these rays, they each have strategies that maintain a stable equilibrium over an extended period. In practice, this means that both players can achieve satisfactory outcomes by following these strategies: the environmental defender efficiently allocates resources to sustain ES, while the polluter minimizes the costs of compensating for harm by adapting to the conditions created by the defender. Thus, Figure 1 illustrates the potential to achieve a sustainable balance of interests, given the parties' strategies and resources, thereby enabling optimal conditions for ES in smart cities.

The computational experiments conducted confirm the effectiveness of the proposed model. The resulting environmental defender's preference sets delineate the boundaries of stable solutions. We also find that, under certain conditions, the defender can reach a steady state in which the polluter is compelled either to reduce emissions or to reallocate resources to circumvent environmental monitoring measures. The visualizations indicate that game-theoretic strategies enable more efficient management of the monitoring-station network and allow adaptive repositioning in response to changing pollution parameters.

The reproducibility of the proposed differential quality game is ensured by the strict formalization of the model via a system of bilinear differential equations and clearly defined preference sets for both players. The employed computational algorithms and numerical schemes allow other researchers to replicate the results for specified initial-condition parameters. The methodology accommodates variability in input data, enabling adaptation of the model to diverse funding scenarios for air-pollution monitoring procedures. The openness of the mathematical formulation and implementation supports independent verification and further extensions of the approach. We believe there is significant potential for further research on applying differential games to assess urban air-quality monitoring resources; in particular, future work could incorporate additional parameters such as temporal changes in pollution levels, seasonality effects, and time delays associated with the introduction of new technologies.

Our results show that a high level of spatial accuracy in monitoring-station placement does not entail increased costs when the proposed strategy is used. This contrasts with a number of other approaches in which placement optimization requires substantial overspending of resources (e.g., [9], [17]). The proposed method benefits from the use of fuzzy sets and adaptive financial strategies of the environmental defender, delivering flexibility without degrading accuracy. Compared with models based solely on hard rules or empirical scenarios, our model exhibits greater adaptability to changing funding conditions in programs for deploying air-quality monitoring stations.

These findings are relevant not only for environmental specialists but also for municipal authorities and urban-planning organizations. The proposed deployment strategy enables more efficient resource management and helps identify critical environmental-protection areas requiring continuous monitoring—particularly useful when designing sustainability programs and setting environmental standards under budget constraints. Moreover, the results can inform evidence-based introduction of ecological fees and subsidies, as well as adjustments to urban transport and industrial policies. For non-governmental organizations and environmental initiatives, the model can serve as a tool for rigorous assessment of environmental conditions and the impacts of various interventions.

Although some researchers have employed the mathematical apparatus of differential games in environmental management—see, for instance, [31]—these studies did not explicitly account for uncertainties inherent to such problems. In later works [31]-[33], a number of researchers attempted to address this issue. Among the relevant studies concerning the application of game-theoretic models to problems of environmental management and resource allocation, several approaches can be identified that are conceptually close to the model proposed in this paper.

Frutos *et al*. [31], analyzed a differential game with spatially distributed control aimed at regulating transboundary pollution, where strategic decisions were influenced by geographic interdependencies between regions. Sarto *et al*. [32], the game-theoretic framework was applied to model pollution control among companies, enabling the formalization of interactions among a large number of agents while considering both pollution dynamics across urban areas and resource allocation for environmental monitoring. Rettieva [33], a setting was examined in which economic and environmental objectives were optimized simultaneously, with pollution levels dependent on the strategies of resource-exploiting players.

However, unlike [31]-[33], where analytical approaches based on the Cauchy formula were applicable, the present model integrates bilinear dynamics and fuzzy information to more adequately reflect real-world eco-protection scenarios. Incorporating incomplete information in the form of fuzziness significantly enhances both the theoretical and practical relevance of the model, as it captures the uncertainty and behavioral variability of the involved stakeholders.

Moreover, the applicability of the proposed game-theoretic model is not limited to a theoretical framework. The problem is closely linked to the advancement of electrotechnical infrastructure and information-measurement systems that enable atmospheric monitoring. The integration of sensor networks based on IoT architectures and the use of modern data transmission technologies (such as LPWAN and 5G) offer the potential for continuous signal acquisition and preliminary data processing using digital filtering and calibration techniques. This technological foundation provides an opportunity for implementing differential game algorithms in real-time environmental monitoring systems.

In the subsequent phase of research, such integration will facilitate the adaptive reallocation of FR between the deployment of new monitoring stations, the modernization of existing sensors, and the introduction of pollution control technologies, based on the current state of the environment. As a result, it becomes possible to align decision-making processes among governmental bodies, industrial enterprises, and civic initiatives—ultimately shaping a comprehensive urban ES policy under increasing anthropogenic pressures.

6.1. Robustness analysis under fuzzy information

Sensitivity tests reveal that ±10% variations in membership function parameters shift the preference set boundaries by approximately 5–8%, while equilibrium strategies remain qualitatively stable. However, when initial states approach critical thresholds (boundaries of W_1), even minor uncertainty can flip the outcome from defender success to polluter advantage. This highlights the practical importance of accurate membership function calibration in real urban monitoring systems.

7. CONCLUSION

A mathematical model of air pollution monitoring in cities is considered. The model, unlike similar ones describing this problem, is built on the assumption that the dynamics of the financial states of the player, both protecting smart cities from air pollution and the player seeking to pollute the air, is specified by a system of bilinear differential equations describing the dynamics of multidimensional variables. It is shown that the controllability of the air pollution monitoring process in cities can be described from the point of view of the game approach based on the solution of a bilinear differential game with several terminal surfaces in a fuzzy formulation. The novelty of the model is that a solution to a bilinear differential quality game with several terminal surfaces in a fuzzy formulation has been found, which adequately reflects the essence of the problem under consideration. The results of a computational experiment are presented. During the computational experiment, performed using a computational module of an IIS implemented in the Python algorithmic language, various parameters ratios describing the air pollution monitoring process in cities were taken into account.

The results presented in the paper may be useful in practice for organizing the ES protection from a party seeking to damage the ES in the case where the defending player does not have crisp information about the financial condition of the polluter.

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AUTHOR CONTRIBUTIONS STATEMENT

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C : **C**onceptualization
M : **M**ethodology
So : **So**ftware
Va : **Va**lidation
Fo : **Fo**rmal analysis
I : **I**nvestigation
R : **R**esources
D : **D**ata Curation
O : Writing - **O**riginal Draft
E : Writing - Review & **E**diting
Vi : **Vi**sualization
Su : **Su**pervision
P : **P**roject administration
Fu : **Fu**nding acquisition

CONFLICT OF INTEREST STATEMENT

Authors state no conflict of interest.

INFORMED CONSENT

We have obtained informed consent from all individuals included in this study.

DATA AVAILABILITY

Data availability does not apply to this article, as no new data were generated or analyzed during the current study.

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