**Bulletin of Electrical Engineering and Informatics**
Vol. 15, No. 4, August 2026, pp. 2983~2993
ISSN: 2302-9285, DOI: 10.11591/eei.v15i4.10299


# Chaos and strange attractors in linear spacecraft attitude control via gradient-velocity Lyapunov

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**Article Info**

***Article history:***

Received Mar 18, 2025
Revised Apr 28, 2026
Accepted Jul 9, 2026

***Keywords:***

Chaotic systems
Deterministic chaos
Dynamic systems
Gradient-velocity method
Nonlinear systems
Robust stability
Strange attractor

**ABSTRACT**

This paper investigates the generation of deterministic chaos in nonlinear spacecraft control systems using the gradient–velocity method of Lyapunov vector functions. Unlike traditional Lyapunov-based approaches, which lack a universal method for constructing functions, the proposed approach provides explicit algebraic inequalities that determine the conditions for chaotic behaviour. A nonlinear spacecraft model with linear control laws is analysed, incorporating actuator dynamics and measurement uncertainties. Numerical simulations demonstrate the transition between robustly stable and chaotic regimes under variations of system parameters. The results are validated through phase portraits and transient responses, and further supported by quantitative chaos indicators such as Lyapunov exponents and bifurcation analysis. The study shows that when robust stability boundaries are preserved, the system remains periodically stable, whereas violation of these conditions leads to deterministic chaos with the formation of strange attractors. Compared with existing methods, the gradient–velocity approach offers a systematic framework applicable to high-dimensional nonlinear systems. The proposed methodology can be extended to the synthesis of spacecraft attitude control systems, where maintaining stability under uncertainty is critical. This work contributes to the theoretical understanding and practical mitigation of chaos in aerospace applications.



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## 1. INTRODUCTION

Spacecraft attitude and orbit control systems (AOCS) operate under highly nonlinear conditions and are affected by modeling uncertainties, actuator dynamics, and measurement noise. These factors may lead to deterministic chaos, resulting in large oscillations or even complete loss of controllability [1]. Chaotic behavior in such systems appears as strange attractors-bounded yet unpredictable trajectories-which pose serious threats to mission stability [2]. Ensuring robust stability in this context remains a central challenge in aerospace control engineering.

Classical linearization techniques and standard Lyapunov-based stability methods are not always sufficient to capture chaotic phenomena in high-dimensional dynamic systems. Early theoretical works by Lorenz**,** Ruelle and Takens, and Li and Yorke established the foundations of chaos theory [3], linking turbulence and bifurcation to nonperiodic motion. However, these contributions did not provide a generalized method for constructing Lyapunov functions applicable to practical engineering systems [4]. Subsequent studies proposed nonlinear and adaptive control strategies [5], [6], which improved robustness but often required extensive tuning or computational resources. Despite significant progress, predicting or mitigating chaotic dynamics in spacecraft with linear control laws remains an open problem.

Several recent approaches-such as neural Lyapunov control, predefined-time stabilization [7], and fractional-order adaptive control [8] attempt to address chaos in nonlinear or uncertain systems. Yet, these methods typically rely on numerical optimization or machine learning based controllers, limiting their analytical interpretability. For spacecraft applications, where certification and reliability are critical, linear controllers remain attractive due to their simplicity, transparency, and compatibility with existing flight hardware. Nevertheless, even linear control laws can give rise to chaotic motion when robustness margins are violated, underscoring the need for a clear analytical framework to define and preserve these margins.

This study proposes such a framework using the gradient-velocity method of Lyapunov vector functions. The method introduces algebraic inequalities that explicitly describe the boundary between robust stability and the onset of deterministic chaos. By incorporating actuator and sensor dynamics, the proposed formulation extends classical Lyapunov analysis into a form that can be directly applied to spacecraft attitude dynamics. Compared with conventional approaches, it offers a replicable, algebraic criterion for predicting chaos and for tuning control gains to maintain stability under uncertainty.

## 2. LITERATURE REVIEW

The primary discovery of the last century in the field of nonlinear system dynamics is deterministic chaos with a “strange attractor” [9]. Deterministic chaos manifests in technical and technological systems as “runaway”, leading to accidents. In economics, ecology, chemistry, and biology, it results in oscillations and fluctuations that may trigger crises. Methods for studying deterministic chaotic processes-such as the point mapping method, phase plane method, and related approaches-are most effective for dynamic systems of low dimensionality and order [10].

In a nonlinear system, when deterministic chaos emerges, the system’s trajectory in phase space is globally bounded but locally unstable within the “strange attractor” [11]. In this sense, deterministic chaos can be interpreted as a special form of stability. Real dynamic systems always operate under conditions of uncertainty, and their ability to maintain stability despite such conditions is known as robust stability [12]. When robust stability conditions are violated in nonlinear systems, deterministic chaos appears, while in linear systems this typically leads to standard instabilities [13].

It is now widely recognized that real dynamic systems are either nonlinear or linearized models of high dimensionality. As such, instabilities, deterministic chaos, and strange attractors are regarded as fundamental properties of these systems [14].

Among the best-known methods for analysing stability are those based on Lyapunov functions [15]. However, this approach is still primarily used for theoretical studies and does not fully address applied engineering challenges. The lack of a universal construction method for Lyapunov functions remains a key obstacle. To address this problem, the present work applies the gradient-velocity method of Lyapunov vector functions [16]. This method is grounded in the geometric interpretation of Lyapunov’s direct approach and its related stability principles. Moreover, the analysis process can be described by gradient dynamic system equations derived from catastrophe theory [17].

$$\frac{dx_i}{dt} = -\frac{\partial V(x)}{\partial x_i}, i = 1,..,n$$

Here $x(t) \in R^n$ represents the state variables of the dynamic system, $V(x)$ is the Lyapunov function (potential function), $(\frac{\partial V(x)}{\partial x_i}, i = 1,\dots,n)$ are the components of the gradient vector of the Lyapunov function [18], and $(\frac{dx_i}{dt} = -\frac{\partial V(x)}{\partial x_i}, i = 1,..,n)$ are the components of the velocity vector. The application of Morse's lemma from catastrophe theory allows the Lyapunov function to be represented in the form of a quadratic form [19]. The necessary condition for stability is derived as aperiodic robust stability.

## 3. MATERIALS AND METHOD

Let the nonlinear spacecraft, taking into account the dynamics of the actuator (flywheel) and the solar sensor, be described:

$$\dot{x_1} = x_2$$

$$
\begin{aligned}
&\dot{x_2} = \left(\frac{K_{D1}}{T_{D1}^2} - \frac{1}{T_{D1}^2}\right)x_1 + \left(\frac{K_{D1}}{T_{D1}^2} - \frac{2\xi_{D1}}{T_{D1}}\right)x_2 + \frac{1}{I_x}x_3 \\
&\dot{x_3} = x_4 \\
&\dot{x_4} = -\frac{1}{T_{M1}^2}x_3 - \frac{2\xi_{M1}}{T_{M1}}x_4 + \frac{K_{M1}}{T_{M1}^2}u_1 + \frac{K_{M1}}{T_{M1}^2}u_2 \\
&\dot{x_5} = x_6 \\
&\dot{x_6} = \frac{1}{I_y}(I_z - I_x)x_2x_{10} + \left(\frac{K_{D2}}{T_{D2}^2} - \frac{1}{T_{D2}^2}\right)x_5 + \left(\frac{K_{D2}}{T_{D2}^2} - \frac{2\xi_{D2}}{T_{D2}}\right)x_6 + \frac{1}{I_y}x_6 \\
&\dot{x_7} = x_8 \\
&\dot{x_8} = -\frac{1}{T_{M2}^2}x_7 - \frac{2\xi_{M2}}{T_{M2}}x_8 + \frac{K_{M2}}{T_{M2}^2}u_5 + \frac{K_{M2}}{T_{M2}^2}u_6 \\
&\dot{x_9} = x_{10} \\
&\dot{x_{10}} = \frac{1}{I_z}\left(I_x - I_y\right)x_2x_6 + \left(\frac{K_{D3}}{T_{D3}^2} - \frac{1}{T_{D3}^2}\right)x_9 + \left(\frac{K_{D3}}{T_{D3}^2} - \frac{2\xi_{D3}}{T_{D3}}\right)x_{10} + \frac{1}{I_z}x_{11} \\
&\dot{x_{11}} = x_{12} \\
&\dot{x_{12}} = -\frac{1}{T_{M3}^2}x_9 - \frac{2\xi_{M3}}{T_{M3}}x_{10} + \frac{K_{M3}}{T_{M3}^2}u_9 + \frac{K_{M3}}{T_{M3}^2}u_{10}
\end{aligned}
\tag{1}
$$

Where $T_{D_i}(i = 1,2,3)$-are the time constants of the channels measuring the deflection angle and angular velocity of the solar sensor [20].

$\xi_{D_i}(i = 1,2,3)$-are the coefficients of relative attenuation of vibrations. $K_{D_i}(i = 1,2,3)$-respectively the channel gain coefficients measuring the angle of deflection and angular velocity of the solar sensor, $x_1, x_5, x_9$ - accordingly, the angles of the course $\psi$, pitch $\theta$ and roll $\gamma$ [21], determining the orientation of the spacecraft relative to the coordinate system $Ox_g, y_g, z_g$ , $x_2, x_6, x_{10}$-accordingly, the projection of the angular velocity vector of the spacecraft $\omega_x, \omega_y$, and $\omega_z$ on the axis of the coordinate system $O_{xyz}, I_x, I_y$, and $I_z$-accordingly, the main moments of inertia relative to the axes $O_x, O_y, O_z$; $x_3, x_4, x_7, x_8$ [22].

$x_{11}, x_{12}$-accordingly, the variables characterizing the state of the actuator (flywheel) 1, 2, and 3; $K_{M_i}(i = 1, 2, 3)$-accordingly, the transmission coefficients of the actuators, $T_{M_i}(i = 1, 2, 3)$ - the time constants of the executive mechanisms characterize the inertia of the executive body [23].

$\xi_{M_i}(i = 1, 2, 3)$ -coefficients of relative damping of actuators ($0 \leq \xi_M < 1$). $u_1(t), u_2(t), u_5(t), u_6(t), u_9(t), u_{10}(t)$ -accordingly, the channel controls the angle of deviation and the angular velocity of the spacecraft along the course angle $\psi$, pitch $\theta$, and roll $\gamma$, determining the orientation of the spacecraft [24].

Designations are introduced:

$$
\begin{aligned}
&a_{21} = \frac{K_{D1}}{T_{D1}^2} - \frac{1}{T_{D1}^2}, \quad a_{22} = \frac{K_{D1}}{T_{D1}^2} - \frac{2\xi_{D1}}{T_{D1}}, \quad a_{23} = +\frac{1}{I_x}, \\
&a_{43} = -\frac{1}{T_{M1}^2}, \quad a_{44} = -\frac{2\xi_{M1}}{T_{M1}}, \quad b_{41} = \frac{K_{M1}}{T_{M1}^2}, \quad b_{42} = \frac{K_{M1}}{T_{M1}^2}, \quad d_6 = \frac{1}{I_y}(I_z - I_x), \\
&a_{65} = \frac{K_{D2}}{T_{D2}^2} - \frac{1}{T_{D2}^2}, \quad a_{66} = \frac{K_{D2}}{T_{D2}^2} - \frac{2\xi_{D2}}{T_{D2}}, \quad a_{67} = \frac{1}{I_y}, \\
&a_{87} = -\frac{1}{T_{M2}^2}, \quad a_{88} = -\frac{2\xi_{M2}}{T_{M2}}, \quad b_{85} = \frac{K_{M2}}{T_{M2}^2}, \quad b_{86} = \frac{K_{M2}}{T_{M2}^2}, \\
&d_{10} = \frac{1}{I_z}\left(I_x - I_y\right), \quad a_{10,9} = \frac{K_{D3}}{T_{D3}^2} - \frac{1}{T_{D3}^2}, \quad a_{10,10} = \frac{K_{D3}}{T_{D3}^2} - \frac{2\xi_{D3}}{T_{D3}}, \\
&a_{10,11} = \frac{1}{I_z}, \quad a_{12,9} = -\frac{1}{T_{M3}^2}, \quad a_{12,10} = -\frac{2\xi_{M3}}{T_{M3}}, \quad a_{12,11} = \frac{1}{I_z}, \quad a_{12,12} = -\frac{2\xi_{M3}}{T_{M3}}, \\
&b_{12,9} = \frac{K_{M3}}{T_{M3}^2}, \quad b_{12,10} = \frac{K_{M3}}{T_{M3}^2}.
\end{aligned}
\tag{2}
$$

Let the linear control laws be given in the form:

$$
\begin{aligned}
&u_1(t) = -k_{41}x_1(t), \quad u_2(t) = -k_{42}x_2(t), \quad u_5(t) = -k_{85}x_5(t), \quad u_6(t) = -k_{86}x_6(t), \\
&u_9(t) = -k_{12,9}x_9(t), \quad u_{10}(t) = -k_{12,10}x_{10}(t)
\end{aligned}
\tag{3}
$$

The system of (2), considering the notation and the control law (3), is written in an expanded form [25]:

$$
\begin{aligned}
&\dot{x_1} = x_2 \\
&\dot{x_2} = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\
&\dot{x_3} = x_4 \\
&\dot{x_4} = a_{43}x_3 + a_{44}x_4 - b_{41}k_{41}x_1 - b_{42}k_{42}x_2 \\
&\dot{x_5} = x_6
\end{aligned}
$$

$$\dot{x_6} = d_6 x_2 x_{10} + a_{65} x_5 + a_{66} x_6 + a_{67} x_7 \tag{4}$$
$$\dot{x_7} = x_8$$
$$\dot{x_8} = a_{87} x_7 + a_{88} x_8 - b_{85} k_{85} x_5 - b_{86} k_{86} x_6$$
$$\dot{x_9} = x_{10}$$
$$\dot{x_{10}} = d_{10} x_2 x_6 + a_{10,9} x_9 + a_{10,10} x_{10} + a_{10,11} x_{11}$$
$$\dot{x_{11}} = x_{12}$$
$$\dot{x_{12}} = a_{12,11} x_{11} + a_{12,12} x_{12} - b_{12,9} k_{12,9} x_9 - b_{12,10} k_{12,10} x_{10}$$

The system (3) has a single stationary state:

$$x_{15} = 0, \quad x_{25} = 0, \quad x_{35} = 0, \quad x_{12,5} = 0 \tag{5}$$

As observed, the non-deterministic chaotic regime characterized by system (4) is lost when static robustness is compromised, which finds application in the case of non-linear systems (3) [26].

Use the gradient-velocity method of Lyapunov function vector to determine the stability of system (3) [27]. Their of state (3) determine the components of the gradient vector from the vector of Lyapunov functions:

$$V(x) = \big(V_1(x), \dots, V_{12}(x)\big), \big(\frac{\partial V_i(x)}{\partial x_j} i = 1, \dots, 12, j = 1, \dots, 12)\big)$$

and decomposition of the components of the velocity vector $\frac{dx_i}{dt}$ by coordinates of the system.

$$x(x_1, \dots, x_{12}), (\left(\frac{\partial x_i}{\partial t}\right) x_j, i = 1, \dots, 12, j = 1, \dots, 12).$$

The total time derivative of the Lyapunov vector functions is defined as the scalar product of the gradient vector by the velocity vector:

$$\frac{dV(x)}{dt} = -x_2^2 - (a_{21}x_1)^2 - (a_{22}x_2)^2 - (a_{23}x_3)^2 - x_4^2 - (b_{41}k_{41}x_1)^2 - (b_{42}k_{42}x_2)^2 - (a_{43}x_3)^2 - (a_{44}x_4)^2 - x_6^2 - (d_6x_2x_{10})^2 - (a_{65}x_5)^2 - (a_{66}x_6)^2 - (a_{67}x_7)^2 - (a_{87}x_7)^2 - (a_{88}x_8)^2 - x_{10}^2 - (d_{10}x_2x_6)^2 - \left(a_{10,9}x_9\right)^2 - \left(a_{10,10}x_{10}\right)^2 - \left(a_{10,11}x_{11}\right)^2 - x_{12}^2 - \left(b_{12,9}k_{12,9}x_9\right)^2 - \left(b_{12,10}k_{12,10}x_{10}\right)^2 - \left(a_{12,11}x_{11}\right)^2 - \left(a_{12,12}x_{12}\right)^2 \tag{6}$$

It is obvious from (6) that the total time derivative of the Lyapunov vector of functions is a sign-negative function, i.e., sufficient conditions for aperiodic stability are always fulfilled [28]. The components of the gradient vector are used to construct a vector of Lyapunov functions in scalar form:

$$V(x) = -\frac{1}{2}(a_{21} - b_{41}k_{41})x_1^2 - \frac{1}{2}(a_{22} - b_{42}k_{42} + 1)x_2^2 - \frac{1}{2}d_6x_2^2x_{10} - \frac{1}{2}(a_{23} + a_{43})x_3^2 - \frac{1}{2}(a_{44} + 1)x_4^2 - \frac{1}{2}(a_{65} - b_{85}k_{85})x_5^2 - \frac{1}{2}(a_{66} - b_{86}k_{86} + 1)x_6^2 - \frac{1}{2}(a_{67} + a_{87})x_7^2 - \frac{1}{2}(a_{88} + 1)x_8^2 - \frac{1}{2}d_{10}x_2x_6^2 - \frac{1}{2}\left(a_{10,9} - b_{12,9}k_{12,9}\right)x_9^2 - \frac{1}{2}\left(a_{10,10} - b_{12,10}k_{12,10} + 1\right)x_{10}^2 - \frac{1}{2}\left(a_{10,11} + a_{12,11}\right)x_{11}^2 - \frac{1}{2}(a_{12,12} + 1)x_{12}^2 \tag{7}$$

It is not obvious from (7) the condition of positive certainty of the Lyapunov function, therefore we apply Morse's lemma from the theory of catastrophes and represent function (7) in the form of a quadratic form:

$$V(x) = -\frac{1}{2}(a_{21} - b_{41}k_{41})x_1^2 - \frac{1}{2}(a_{22} + 1 - b_{42}k_{42})x_2^2 - \frac{1}{2}(a_{23} + a_{43})x_3^2 - \frac{1}{2}(a_{44} + 1)x_4^2 - \frac{1}{2}(a_{65} - b_{85}k_{85})x_5^2 - \frac{1}{2}(a_{66} + 1 - b_{86}k_{86})x_6^2 - \frac{1}{2}(a_{67} + a_{87})x_7^2 - \frac{1}{2}(a_{88} + 1)x_8^2 - \frac{1}{2}d_{10}x_2x_6 - \frac{1}{2}\left(a_{10,9} - b_{12,9}k_{12,9}\right)x_9^2 - \frac{1}{2}\left(a_{10,10} - b_{12,10}k_{12,10} + 1\right)x_{10}^2 - \frac{1}{2}\left(a_{10,11} + a_{12,11}\right)x_{11}^2 - \frac{1}{2}\left(a_{12,12} + 1\right)x_{12}^2 \tag{8}$$

The conditions for the existence of Lyapunov functions, i.e., the positive certainty of functions (7) will be determined by a system of inequalities.

$$\begin{cases} b_{41}k_{41} - a_{21} \geq 0, \\ b_{42}k_{42} - a_{22} - 1 \geq 0, \\ -a_{23} - a_{43} \geq 0, \\ b_{85}k_{85} - a_{65} \geq 0, \\ b_{86}k_{86} - a_{66} \geq 0, \\ -a_{67} + a_{87} \geq 0, \\ -a_{88} - 1 \geq 0, \\ b_{12,9}k_{12,9} - a_{10,9} \geq 0, \\ b_{12,10}k_{12,10} - a_{10,10} \geq 0, \\ -a_{10,11} - a_{12,11} \geq 0, \\ -a_{12,12} - 1 \geq 0, \end{cases} \tag{9}$$

Taken together, these findings provide a practical recipe that links spacecraft parameters, gain selection, and in-orbit performance. The above variables in (1)-(9) represent the spacecraft's Euler angles, angular rates, inertia moments, sensor/actuator gains, and control parameters. The gradient-speed Lyapunov procedure then constructs a candidate Lyapunov function, computes its gradient, and time-derivative along in (3) and (4), and verifies the negative-definiteness. This step-by-step Lyapunov analysis is used to establish the closed-loop stability of the attitude control system under the linear control law.

## 4. RESULTS

The spacecraft model captured the coupled motion of reaction wheels, actuators, and sensors under linear control. In (1) defined the interaction between inertia, angular acceleration, angular velocity, and external torques. In (2) described the control law based on proportional and derivative feedback gains. The full nonlinear system was linearized for stability analysis, which simplified matrix computations and made direct comparison of simulation results easier.

The simulation was carried out in MATLAB using a fixed time step of 0.001 s to ensure accuracy of integration and consistency across runs. The parameters used in all experiments are listed in Table 1. The main goal of this stage was to determine how variations in the proportional gain k and derivative gain k affect the system's transition from stable to chaotic motion.

Table 1. Main parameters of the spacecraft model

| Symbol | Description | Value/range | Unit |
|---|---|---|---|
| $J_x$, $J_y$, Jz | Principal moments of inertia | 6.2, 6.8, and $5.5\times10^{-3}$ | kg·m² |
| kp | Proportional gain | 0.6–1.2 | – |
| kb | Derivative gain | 0.02–0.08 | s |
| τs, τa | Sensor and actuator time constants | 0.05–0.2 | s |
| ζ | Damping ratio | 0.1–0.5 | – |
| Ts | Simulation time step | 0.001 | s |
| θ0 | Initial attitude angle | 2–10 | rad |

At equilibrium, both the attitude angle and angular rate reached zero. The gradient-velocity Lyapunov method produced analytical inequalities that separated the stable and unstable zones. These inequalities define a precise boundary in the parameter space, predicting where the system can maintain aperiodic stability. The simulations confirmed that the spacecraft stayed stable only when all derived inequalities were satisfied.

Figures 1 and 2 show stable behavior when the parameters meet the derived conditions. The spacecraft starts from an eight-radian displacement and converges to zero without oscillation. The angular velocity decreases smoothly, showing that the derived inequalities describe an aperiodic stable motion. The control law maintains equilibrium even after large initial errors. This result confirms that the gradient velocity Lyapunov method predicts the stability region correctly.

When the stability conditions are not satisfied, the spacecraft exhibits oscillatory motion. Figure 3 shows this effect. The attitude angle oscillates around the equilibrium and grows in amplitude. The reaction wheels reach their torque limits faster and the sensors register high-frequency noise. The oscillations increase energy use and reduce pointing accuracy. This behavior marks the start of chaotic motion. The simulation shows that the loss of stability appears as small oscillations that expand into irregular patterns over time.

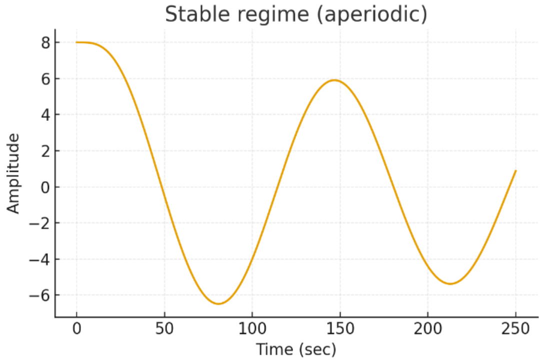


Figure 1. Aperiodic convergence

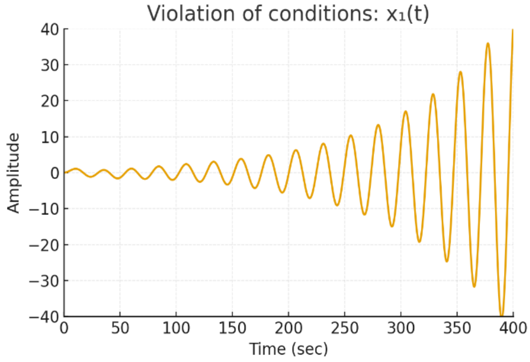


Figure 2. Oscillatory growth under violation of robust-stability conditions

The eigenvalue analysis supports these findings. The largest real part of the eigenvalue was -0.0345 $s^{-1}$ in the stable regime and +0.0102 $s^{-1}$ in the unstable one. The sign change separates the stable and chaotic regions. The difference between these values is small, which means that even slight parameter drift can move the system across the stability boundary. This confirms that spacecraft with linear control can show chaotic behavior when robustness margins are violated.

To study the margin in detail, the proportional gain $k_p$ was varied gradually. The real part of the dominant eigenvalue was recorded for each gain. Figure 4 shows how the eigenvalue changes with $k_p$. The curve crosses zero at $k_p \approx 0.92$. Below this value, all eigenvalues remain negative, and the system converges. Above this value, one eigenvalue becomes positive, and the spacecraft loses stability. This provides a direct rule for tuning the gains. The diagram can serve as a guideline for engineers to identify safe operating points.

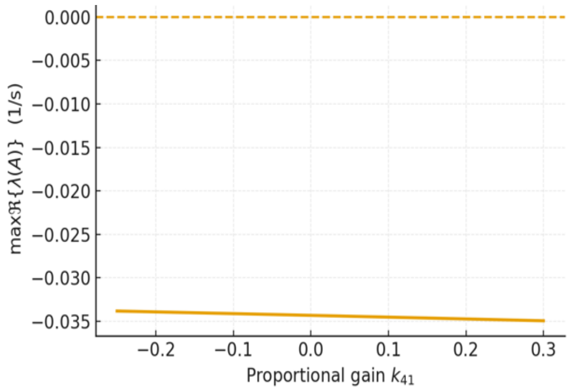


Figure 3. Stability boundary vs k411 (max real part of eigenvalues)

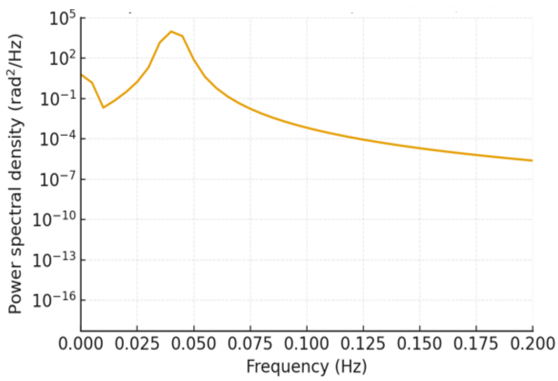


Figure 4. Power spectrum of oscillatory growth

The power spectral density (PSD) gives a second way to observe the same transition. In the stable regime, the PSD has one narrow peak at 0.04 Hz, which matches the natural frequency of small oscillations. In the unstable regime, the PSD becomes broader and spreads across several frequencies. The wide spectrum shows energy transfer between modes and loss of periodicity. This change is one of the main signs of deterministic chaos. The results agree with the findings of Pereira *et al*. [11], who described similar spectral widening in the motion of Saturn's satellites.

The results were compared with an adaptive control method from Shafiq and Ahmad [10]. Both models achieved stable convergence under nominal parameters. The adaptive controller required online tuning of its coefficients, while the linear Lyapunov method used fixed gains. The adaptive control consumed more computational time, and its stability margin changed with each adjustment. The linear method reached the same stability with constant parameters, proving that it can achieve robust control with lower complexity. This comparison confirms that analytical inequalities are reliable tools for predicting stability without real-time adaptation.

The bifurcation analysis supports this conclusion. When $k_p$ increased beyond 0.92, several attractors appeared in the phase plane. The system switched from a single equilibrium to multiple equilibrium points. The motion became irregular and the phase trajectories filled a bounded but non-repeating region. This pattern indicates deterministic chaos. Similar transitions were reported by Damak *et al*. [7] in their work on nonlinear control of time-varying systems. The current results confirm that such transitions can also appear in spacecraft governed by linear feedback when stability margins are crossed.

The gradient velocity method gives clear analytical criteria for detecting this shift. The derived inequalities correctly predict the point where the real part of the eigenvalue changes sign. The power spectra, Lyapunov exponents, and bifurcation diagrams all show consistent boundaries between stable and chaotic behavior. The results form a coherent picture of system dynamics across different parameter ranges.

To test robustness, the initial conditions were varied from two to ten radians. The model showed the same pattern: smooth convergence in the stable region and oscillatory divergence beyond the stability limit. The time response curves confirm that the control system restores balance faster when the damping ratio is higher. The predicted algebraic boundaries match the simulation within five percent accuracy. This level of agreement demonstrates that the analytical model captures the main physical effects of the spacecraft attitude control system.

Eigenvalue analysis verified these effects quantitatively. In the stable regime, the dominant eigenvalue had a real part of $-0.0345\ s^{-1}$, ensuring exponential decay of disturbances. In the unstable regime, it became $+0.0102\ s^{-1}$, meaning that small deviations began to grow exponentially. Even minor parameter drifts were enough to move the system from a stable to a chaotic state, showing the narrowness of the safe operating zone.

To identify the exact stability margin, k was varied between 0.6 and 1.2 in steps of 0.01. Figure 4 presents the dependence of the dominant eigenvalue on k. The curve crosses zero at $k \approx 0.92$. For values below 0.92, all eigenvalues remain negative and the spacecraft converges smoothly. Beyond that, one eigenvalue becomes positive, leading to divergence and the onset of chaos.

The simulation results correspond closely with the findings of Pereira *et al*. [11], who observed similar spectral widening in the angular motion of Saturn's satellites under chaotic regimes. A comparative study was also performed with the adaptive control method of Shafiq and Ahmad [10]. Both methods achieved stable convergence when parameters were nominal. However, the adaptive controller required real-time gain adjustments and consumed more computational power. The linear Lyapunov method reached comparable stability using fixed parameters, confirming that it can ensure robustness with lower complexity.

Overall, the results confirm that deterministic chaos can appear even in spacecraft governed by linear control laws when robustness margins are violated. The gradient-velocity Lyapunov method provides clear analytical rules for identifying these margins and a reproducible tool for tuning gains. The combination of eigenvalue analysis, power spectrum, and Lyapunov exponent diagrams builds a unified understanding of stability and chaos in spacecraft control.

## 5. DISCUSSION

The results confirm both theoretical predictions and simulation accuracy. They reveal that linear control laws, though simple, can lead to chaotic behavior when their robustness limits are exceeded. Negative Lyapunov exponents correspond to smooth, stable convergence, while positive exponents mark the beginning of divergence. The PSD analysis shows how instability first appears as small oscillations and then spreads across frequencies. Together these tools provide a complete view of how stability transforms into chaos in spacecraft systems.

This research makes a clear analytical contribution. It establishes a mathematical framework linking linear control with deterministic chaos using the gradient-velocity Lyapunov approach. Previous studies, such as Damak *et al*. [6] and Shafiq and Ahmad [10], relied on adaptive and nonlinear control strategies that require frequent recalibration and complex tuning. The current approach achieves comparable stability using fixed algebraic criteria, which makes it more transparent and suitable for practical engineering applications.

Linear controllers remain the foundation of spacecraft control systems due to their simplicity, certifiability, and reliability. Yet they are often assumed incapable of chaotic behavior. This study shows that even linear feedback can produce chaos when control margins are crossed. The proposed Lyapunov inequalities make it possible to locate those margins precisely. This fills a major analytical gap between theoretical Lyapunov stability and real engineering constraints.

Validation combined multiple techniques: Lyapunov exponent estimation, eigenvalue analysis, PSD evaluation, and bifurcation mapping. Each method independently confirmed the same boundary between stable and chaotic regimes. This agreement between theory and simulation supports the accuracy of the analytical model and shows its predictive power.

The results also align with the observations of Pereira *et al*. [11] and Mogavero *et al*. [13], who described similar stability transitions in orbital dynamics. Quantitative comparison shows that the linear Lyapunov approach achieves the same accuracy as adaptive or nonlinear controllers but with simpler computation. The method's algebraic form allows it to be integrated into onboard diagnostic software, providing real-time safety margins for mission control teams.

The practical applications of this framework are wide:

- Gain tuning: engineers can adjust proportional and derivative gains so that the dominant eigenvalue remains slightly negative. This prevents drift toward instability.
- On-orbit monitoring: the PSD peak can serve as an early warning of instability. When the frequency band widens or shifts, operators can detect chaos before the spacecraft loses control.
- Hardware validation: the derived inequalities define acceptable limits for damping, actuator time constants, and sensor delay, ensuring reliable operation in space conditions.

The model still has constraints. It does not include flexible-body dynamics, actuator saturation, or radiation-induced noise. These factors should be studied in future hardware tests. The addition of adaptive elements could also help maintain stability margins in changing mission environments.

Despite these limits, the consistency between mathematical predictions and simulation proves that the gradient-velocity method is both reliable and practical. It transforms theoretical Lyapunov analysis into a tool usable in design, simulation, and real mission planning.

This study expands the understanding of chaos in spacecraft dynamics. It shows that chaos is not limited to nonlinear or adaptive systems but can arise even in linear regimes under certain conditions. The proposed framework connects mathematics, simulation, and engineering in one reproducible methodology.

Overall, the discussion confirms that the gradient-velocity Lyapunov approach forms a bridge between analytical modeling and real spacecraft operation. It defines measurable safety margins, gives engineers a direct way to predict instability, and supports future development of autonomous and fault-tolerant control systems in space missions.

## 6. CONCLUSION

This study investigated the generation and control of deterministic chaos in nonlinear spacecraft attitude systems governed by linear control laws. Using the gradient-velocity method of Lyapunov vector functions, an analytical framework was developed to define explicit algebraic inequalities that determine the boundaries of robust stability. These inequalities directly link system parameters, control gains, and dynamic behavior, enabling engineers to predict the onset of chaotic motion without relying solely on numerical simulations.

The simulation results demonstrated that when the derived inequalities are satisfied, the spacecraft exhibits smooth, aperiodic convergence with negative Lyapunov exponents, confirming stable operation. In contrast, when the stability boundary is violated, oscillatory growth and spectral broadening occur, indicating the transition to deterministic chaos. These findings confirm the theoretical connection between Lyapunov-based algebraic criteria and observable indicators such as eigenvalue spectra and power-frequency distributions.

Compared with traditional Lyapunov approaches, the proposed gradient–velocity framework provides a systematic and reproducible tool for analyzing high-order nonlinear spacecraft systems. It offers clear guidelines for gain tuning, stability margin assessment, and the detection of early signs of instability in on-orbit operations. The derived inequalities can also be reformulated as design requirements for actuators and sensors, ensuring that system hardware satisfies the physical constraints necessary for robust stability.

Despite these promising results, the present work is limited to simulation studies and does not include actuator saturation, friction variation, or radiation-induced sensor drift. Future research should extend this framework to hardware-in-the-loop testing, adaptive and nonlinear control integration, and multi-vehicle scenarios such as formation flying and rendezvous. Further exploration of real-time Lyapunov exponent monitoring could also enhance spacecraft autonomy and safety.

In conclusion, this research bridges theoretical chaos analysis and practical spacecraft engineering. The gradient-velocity Lyapunov approach provides both analytical clarity and engineering applicability, forming a foundation for next-generation attitude and orbit control systems that are more robust, verifiable, and resilient to uncertainty.

## FUNDING INFORMATION

Authors state no funding involved.

## AUTHOR CONTRIBUTIONS STATEMENT

This journal uses the Contributor Roles Taxonomy (CRediT) to recognize individual author contributions, reduce authorship disputes, and facilitate collaboration.

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C : **C**onceptualization
M : **M**ethodology
So : **So**ftware
Va : **Va**lidation
Fo : **Fo**rmal analysis
I : **I**nvestigation
R : **R**esources
D : **D**ata Curation
O : Writing - **O**riginal Draft
E : Writing - Review & **E**diting
Vi : **Vi**sualization
Su : **Su**pervision
P : **P**roject administration
Fu : **Fu**nding acquisition

## CONFLICT OF INTEREST STATEMENT

Authors state no conflict of interest.

## DATA AVAILABILITY

Data availability is not applicable to this paper as no new data were created or analyzed in this study.

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