Comparative analysis of observer-based LQR and LMI controllers of an inverted pendulum

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ABSTRACT

An inverted pendulum is a multivariable, unstable, nonlinear system that is used as a yardstick in control engineering laboratories to study, verify and confirm innovative control techniques. To implement a simple control algorithm, achieve upright stabilization and precise tracking control under external disturbances constitutes a serious challenge. Observer-based linear quadratic regulator (LQR) controller and linear matrix inequality (LMI) are proposed for the upright stabilization of the system. Simulation studies are performed using step input magnitude, and the results are analyzed. Time response specifications, integral square error (ISE), integral absolute error (IAE) and mean absolute error (MAE) were employed to investigate the performances of the proposed controllers. Based on the comparative analysis, the upright stabilization of the pendulum was achieved within the shortest possible time with both controllers however, the LMI controller exhibits better performances in both stabilization and robustness. Moreover, the LMI control scheme is effective and simple.

Keywords: Inverted pendulum, Linear matrix inequality, Linear quadratic regulator, Nonlinear system, Observer, Robustness

1. INTRODUCTION

Control of an inverted pendulum using various control strategies has become a topic of interest for many years owing to being an under-actuated, unstable, multivariable and non-linear. Based on the literature, the following are some of the control strategies employed for stabilization and tracking control of an inverted pendulum system; a variable speed control moment gyroscope (VSCMG) actuator was employed to control a typical pendulum configuration [1]. Coupled state-dependent Riccati equation (SDRE) method for scientifically designing nonlinear quadratic regulator (NLQR) and H infinity control of an under-actuated Furuta rotary pendulum was developed [2]. The main control objectives of a rotary inverted pendulum as swing-up control, stabilization control, switching control and trajectory tracking control was described in [3, 4] proposed RBF-ARX (state-dependent auto-regressive model with exogenous input and radial basis function network type coefficients) model-based efficient robust predictive control method for systematic inverted pendulum design. According to [5] the output feedback laws with a minimum switching rule for saturated
switched linear systems were developed, and it provided control synthesis conditions of a spherical inverted pendulum. Improvement of Polekhin’s theorem by lowering the regularity motion and a periodic solution for the carriage moving periodically on the plane was obtained in [6]. An open-source online laboratory experiment was offered by [7] for the Furuta pendulum, a system that has helped researchers to study nonlinear dynamics and control theory. An automatic self-tuning control system were proposed for inverted pendulum system via LQR controller, precise optimal stabilization was achieved [8]. A feedback linearization and sliding mode control approach were established to stabilize a class of fourth-order nonlinear systems where design parameters of the sliding surface were modified using the adaptation laws, based on the gradient descent technique [9].

However, another approach by [10], stabilized a wheeled inverted pendulum using one accelerometer with a modified mechanical structure. Similarly, approximate feedback linearization and sliding mode methods were employed to control a cart-type inverted pendulum, where stability was achieved by using an optimized hybrid algorithm based on the particle swarm optimization and genetic algorithm [11]. Linear control of the flywheel inverted pendulum was proposed in [12], in which high performances were achieved. In a different control approach, [13] controlled reaction wheel oscillation using a proportional-integral controller (PID), while a linear tracking controller was developed for velocity control of a two-wheeled inverted pendulum (TWIP) mobile robot based on its Takagi-Sugino (T-S) fuzzy model [14]. An investigation was carried out by [15] where the inverted pendulum is used and a real interpolation method is employed. The adaptive neuro-fuzzy inference system (ANFIS) was introduced to control an inverted pendulum system, where the desire position was tracked [16]. A comparison between an LQR and pole placement was presented for the stabilizing cart pendulum [17]. An algorithm in [18] was developed for single input multi output with under-actuated systems with mismatched uncertainties. In the pole-placement method, the closed-loop pole location must be determined. But the researcher may know where they are located. The optimal control method ignores finding the desired pole location, the control law of the optimal control method always optimizes the performance of the system to avoid drawbacks.

This paper presents a comparison of observer-based LQR and LMI controllers for stabilization control of an inverted pendulum system. The major problem is the pendulum Stabilization in an upright position and stoppage of cart movement at the desired location within a short time. Assessment of LMI and LQR control algorithms shows several advantages that improved closed-loop stability. Section 2 presents system describe, section 3 presents the system dynamic model and control schemes. Section 4 present results and discussion while section 5 conclusions and future recommendation.

2. SYSTEM DESCRIPTION

The system model is 33-000-V73 which consists of dual pendula, cart, D.C motor, and a rail. For the pendula to rotate freely through $360^\circ$, the two are hinged at the cart’s centre [19]. The D.C motor moves the cart horizontally on the rail, freely. Figure 1 shows the mechanical system and the system is unstable whenever the pendula are positioned vertically but downright stable when positioned downwardly. Any slight deviation from the equilibrium point would render the pendulum unstable. The moving region of the inverted pendulum in which control is achievable has been shown in Figure 2.
2.1. System modelling

Figure 3 shows the system representations, where \( x, f(t), \) and \( \theta \) are the cart displacement, the applied force (N) and the pendulum angle respectively. \( I \) is the moment of inertia (kg/m\(^2\)) of the rod from the centre of mass, \( M \) is the cart’s mass (kg) and \( l \) is the length (m) of the pendulum. The constants, \( b, \) and \( c \) are the cart’s viscous and translation damping (Ns/m) respectively. The system parameters are as recorded in Table 1 [19].

![Figure 3. Schematic diagram](image)

Based on Figure 3, the system’s overall dynamic equations are obtained as:

\[
\begin{cases}
(M + m)\ddot{x} + c\dot{x} + m\cos\theta\dot{\theta} + ml\sin\theta\dot{\theta}^2 = f(t) \\
(ml^2 + l)\ddot{\theta} + ml\dot{x}\cos\theta + mgl\sin\theta + b\dot{\theta} = 0
\end{cases}
\]

(1)

The dynamic equation can, however, be represented in a state-space form as:

\[
\dot{x} = Ax + Bu
\]

(2)

\[
y = Cx
\]

(3)

and the states vector of the system expressed as:

\[
Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}
\]

(4)

where \( \theta, \dot{\theta}, x \) and \( \dot{x} \) are the pendulum angle, angular velocity, cart displacement and velocity of the cart respectively. The system matrix was obtained as [19]:

\[
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.263 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.3358 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 7.0501 \\ 0 \\ 10.6649 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

3. CONTROL SYSTEM DESIGN

3.1. LQR control

In the LQR control system, a control law is selected to regulate the state \( x \) and to get the performance index minimized:

\[
J = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)
\]

(5)

where \( J \) is the performance index, \( R > 0 \) and \( Q \geq 0 \) are the weight matrices for the control variable \( u(t) \) and state variable \( x(t) \) respectively. Figure 4 shows a typical LQR control system.

Table 1. System parameters [19]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of the cart (M)</td>
<td>2.4 kg</td>
</tr>
<tr>
<td>Mass of pole (m)</td>
<td>0.23 kg</td>
</tr>
<tr>
<td>Length of pole (l)</td>
<td>0.38 m</td>
</tr>
<tr>
<td>Moment of inertia of the pole (I)</td>
<td>0.099 kg/m(^2)</td>
</tr>
<tr>
<td>Coefficient of friction of cart (b)</td>
<td>0.05</td>
</tr>
<tr>
<td>Damping coefficient of pendulum (d)</td>
<td>0.005 Nms/rad</td>
</tr>
<tr>
<td>Gravity (g)</td>
<td>9.8 m/s(^2)</td>
</tr>
</tbody>
</table>
Comparative analysis of observer-based LQR and LMI controllers of an inverted...

(Nura Musa Tahir)

**Figure 4. Typical LQR control system**

\[ R \text{ and } Q \text{ are positive definite and the semi-positive definite matrices respectively. Thus, } K \text{ can be obtained to satisfy the feedback control law [20-22]:} \]

\[ u = -Kx = -R^{-1}B^TPx \]  

(6)

where; \( P \) is the solution of the Riccati equation:

\[ A^TP + PA + Q - PB R^{-1}B^TP = 0 \]  

(7)

\[ K = R^{-1}B^TP \]  

(8)

The closed-loop controller gains were found as \( K = [20.9777; 8.3690; -3.9528; -4.1689] \).

**3.2. Observer feedback control design**

The estimator poles were chosen ten times as faster as the system poles and the control law with the observer design is combined to get a compensator as shown in Figure 5 [23, 24]. The observer gain \( L \) was obtained using MATLAB routine called “place”. The observer pole was assigned as \( P \). Therefore, the observer gain is obtained using equation (12) as:

\[ L = \text{place}(A', C', P)' \]  

(9)

This generated an observer gain that placed the poles in the desired position. The following parameters were used in the design of the observer in MATLAB. The weight due to angle and position values are \( Q = \text{diag} \{25, 0, 250, 0\} \) and \( R=1.6 \). Thus, the observer gain and the closed-loop system gain was obtained as; \( K = [55.5439; 22.2266; -12.5000; -12.0038] \) and

\[
\begin{bmatrix}
0.0106 & -0.0708 \\
0.0415 & -1.3843 \\
0.1393 & 0.1695 \\
3.6546 & 3.8795
\end{bmatrix}
\]

**Figure 5. Observer feedback control system**

**3.3. Proposed LMI controller design**

Stability and transient response of linear systems depend on the locations of the poles in the complex plane. Consider a linear dynamic system;

\[ \dot{X} = A_cX \]  

(10)

The Lyapunov theorem states that the system in equation (10) is said to be asymptotically stable once a real symmetric matrix \( P \) satisfying the following LMIs exists [25, 26]:

\[
\begin{bmatrix}
R & Q \\
Q^T & P
\end{bmatrix}
\begin{bmatrix}
A_c & B_c \\
B_c^T & K
\end{bmatrix}
\begin{bmatrix}
R & Q \\
Q^T & P
\end{bmatrix}
\]

\[ \leq 0 \]
\[ A_cP + A_cA_c^T < 0, \quad P = P^T > 0 \] (11)

The LMIs in (11) provides the conditions for the stability of the system in (10). The left half plane and LMI region is as presented in Figures 6 and 7.

All the poles of the system in (10) will be lying in the LMI region of Figure 7 if and only if there exists asymmetrical positive definite matrix \( P \) such that:

\[
A_cP + PA_c^T + 2aP < 0, \tag{12}
\]
\[
A_cP + PA_c^T + 2bP > 0, \tag{13}
\]
\[
\begin{bmatrix}
-rP & cP + PA_c^T \\
cP + A_cP & -rP
\end{bmatrix} < 0, \quad P = P^T > 0 \tag{14}
\]

The LMIs in (12) and (13) represent the vertical strip, while the LMI in (14) represents the circle centred at \((c, 0)\) with radius \(r > 0\). The controller gains \( K \) and the reference input scaling factor \( N \) were found as \([20.9777; 8.3690; -3.9528; -4.1689]\) and \([-461.5637; -178.3047; 155.2348; 104.0856]\) respectably by using MATLAB codes.

4. RESULTS AND DISCUSSION

In this section, using the step input in MATLAB software the proposed control schemes were implemented. Figure 8 shows the open-loop response and Table 2 recorded the frequency, eigenvalues, and the damping ratio of the open-loop system which confirms the system is completely unstable. The system is completely stable after applying the observer-based LQR controller as shown in Figure 9, whereby all poles of the system were located in the negative half-plane. This can also be confirmed based on the recorded data of the eigenvalues, damping and frequency as in Table 3.

<table>
<thead>
<tr>
<th>Frequency (rad/s)</th>
<th>Eigenvalue</th>
<th>Damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00e+000</td>
<td>0.00e+000</td>
<td>-1.00e+000</td>
</tr>
<tr>
<td>0.00e+000</td>
<td>0.00e+000</td>
<td>-1.00e+000</td>
</tr>
<tr>
<td>2.52e+000</td>
<td>2.52e+000</td>
<td>-1.00e+000</td>
</tr>
<tr>
<td>2.52e+000</td>
<td>2.52e+000</td>
<td>-1.00e+000</td>
</tr>
</tbody>
</table>

Table 2. Open-loop simulation data
Comparative analysis of observer-based LQR and LMI controllers of an inverted… (Nura Musa Tahir)

Table 3. Closed-loop simulation data

<table>
<thead>
<tr>
<th>Frequency (rad/s)</th>
<th>Eigenvalue</th>
<th>Damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00e+001</td>
<td>-7.15e+000 + 7.01e+000i</td>
<td>7.14e-001</td>
</tr>
<tr>
<td>1.00e+001</td>
<td>-7.15e+000 + 7.01e+000i</td>
<td>7.14e-001</td>
</tr>
<tr>
<td>2.28e+000</td>
<td>-2.18e+000 + 6.56e-001i</td>
<td>9.58e-001</td>
</tr>
<tr>
<td>1.00e+002</td>
<td>-7.15e+000 + 7.01e+001i</td>
<td>7.14e-001</td>
</tr>
<tr>
<td>1.00e+002</td>
<td>-7.15e+000 + 7.01e+001i</td>
<td>7.14e-001</td>
</tr>
<tr>
<td>2.28e+001</td>
<td>-2.18e+001 + 6.56e-000i</td>
<td>9.58e-001</td>
</tr>
<tr>
<td>2.28e+001</td>
<td>-2.18e+001 + 6.56e-000i</td>
<td>9.58e-001</td>
</tr>
</tbody>
</table>

Table 4. LMI closed-loop simulation data

<table>
<thead>
<tr>
<th>Frequency (rad/s)</th>
<th>Eigenvalue</th>
<th>Damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.46e+00</td>
<td>-3.88e+000 + 3.84e+000i</td>
<td>7.11e-01</td>
</tr>
<tr>
<td>5.46e+0</td>
<td>-3.88e+000 - 3.84e+000i</td>
<td>7.11e-01</td>
</tr>
<tr>
<td>1.76e+00</td>
<td>-1.73e+000 + 2.82e-001i</td>
<td>9.87e-01</td>
</tr>
<tr>
<td>1.76e+00</td>
<td>-1.73e+000 - 2.82e-001i</td>
<td>9.87e-01</td>
</tr>
</tbody>
</table>

Figure 9. Observer close loop poles and zeros map

The cart position and swing angle of the system were as shown in Figure 10, simulated with an initial condition of 0.1rad. The systems stabilized at 2.4 sec with 0.262 undershoot of cart position and 0.07 of the swing angles. Moreover, observer-based state feedback was designed to estimate the system output and generate the control signal that yields the desired closed-loop performance. The tick poles are the observer poles while the star-like poles are the system poles. This implies that the observer poles are more negative away from system poles. Thus, the closed-loop poles of the observed states’ feedback system have system poles and the observer poles. They were designed separately and combined to form an observer feedback control system. Therefore, the observer poles are chosen in such a way that the observer response is much faster than the system response so that the observer has less effect on the system.

Besides, the LMI controller was designed and compared with observer-based LQR. In this control schemes, the closed-loop poles need to be placed in the LMI region of the complex left-half plane to achieve a stabilized system and good transient response. In designing LMI based controller, the transient parameters were selected as; r=200 which is the radio of the circle, c=100 which is the centre of the circle and a=3 and b=200 which are two points on the circle. The best value of $t$ should be negative for feasibility, thus obtained as -3.669348e-05 and f-radius saturation is 0.000% of $R$ which is equal to 1.00e+09, and Table 4 recorded the simulation data of the LMI control algorithms.

Figure 10. Closed loop response of the system

The observer-based LQR controller does its job of maintaining the pendulum angle in an upright position, but its response is sluggish as compared to the LMI based controller. The controller’s performances as in Figures 10, 11, and Table 5 show clearly that LMI controller is superior over observer-based LQR controller. Also, integral square error, integral absolute error and mean absolute error were used as the performance indexes, and based on the recorded data as shown in Table 5, both controller performances excellently as they recorded less error. However, LMI based controller shows a better performance.
5. CONCLUSION

In this paper, the performances of the observer-based LQR and LMI controllers were investigated for upright stabilization control of nonlinear inverted pendulum. Observer-based LQR and LMI controllers were designed to stabilize the pendula. Time response specifications, integral absolute error, integral square error and mean absolute error were employed to investigate the performances of the proposed controllers. Based on simulation results and performance index analysis, a better performance was achieved using the LMI controller. To further reduce the amplitude and frequencies of the oscillation, frictional coefficients should be taken into consideration.

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REFERENCES


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