

An optimistic-pessimistic game cross-efficiency method based on a Gibbs entropy model for ranking decision making units

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Article Info

Article history:

Received Jan 11, 2023

Revised Jul 7, 2023

Accepted Oct 9, 2023

Keywords:

Data development analysis
Game cross-efficiency method
Gibbs entropy
Gibbs entropy linear
programming model
Optimistic-pessimistic game
cross-efficiency method

ABSTRACT

The game cross-efficiency method, a commonly utilized approach for ranking decision-making units in tie-breaking scenarios, is based on secondary goals. However, in certain data envelopment analysis ranking problems, the classical game cross-efficiency method may fail to differentiate all decision-making units effectively. To address this limitation, it is prudent to explore the development of a new method that can enhance the ranking performance of the classical game cross-efficiency approach. In this study, we propose a novel Gibbs entropy linear programming model that integrates both optimistic and pessimistic perspectives of the classical game cross-efficiency method for data envelopment analysis ranking problems. To validate the reliability and utility of our proposed method, we present three examples: the six nursing homes problem, numerical example 2, and an application involving twenty Thai provinces with cash crop data. The reliability of the proposed method is assessed using Spearman's correlation coefficient (r_s) on the numerical examples. The results demonstrate that the r_s values for both the proposed method and the classical game cross-efficiency method, specifically for the six nursing homes problem, numerical example 2, and the application involving twenty Thai provinces, are determined to be $r_s=0.998$, 0.998, and 0.986 respectively.

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1. INTRODUCTION

Initially, Farrell [1] introduced a methodology to assess the performance of a group of comparable decision-making units (DMUs) characterized by multiple inputs and outputs. However, Charnes *et al.* [2] who were credited with being the first to operationalize Farrell [1] concept into a data envelopment analysis model data envelopment analysis (DEA) model, famously known as the Charnes, Cooper and Rhodes (CCR) model, named in honor of the three authors' initials. The DEA model computes a DMU's maximum relative efficiency score via a linear programming model that seeks to maximize the output-to-input ratio. Typically, when a DMU achieves a relative efficiency score of 1, it is classified as efficient. Given its proven effectiveness and versatility, the DEA model has garnered significant attention and found applications across diverse domains. Banker *et al.* [3] introduced the Banker, Charnes and Cooper (BCC) model, a widely employed framework with applications spanning various fields, including economics and finance. The DEA

model's performance measurements have the advantage of evaluating the performance of many DMUs without the need to standardize input and output data. It is unnecessary to assume the production function assumptions for DEA. In addition, there is no requirement to establish input and output weights because the weight may be generated using DEA model [4], [5]. DMUs can be compared to manufacturing units, businesses, schools, banks, hospitals, universities, and commercial firms in this context. The DEA can classify DMUs into two distinct categories: efficient and inefficient. The efficient DMUs can create the same output or more with fewer inputs, whereas the inefficient DMUs require more inputs to produce the same result or less. Nevertheless, the DEA model lacks the capability to rank efficient DMUs (relative efficiency score values of 1) [6]–[8]; therefore, Sexton *et al.* [9] enhanced this methodology and presented the DEA cross-efficiency method.

The cross-efficiency approach combines self-assessment and peer-evaluation to determine the relative efficiency of each DMU. This technique offers several key advantages [10]–[12]. Firstly, it effectively discriminates between strong and weak performance, resulting in a comprehensive DMU ranking. Secondly, it overcomes the challenge of unrealistic weight schemes without the need for weight constraints. By leveraging assessed weights from DMUs and other DMUs, the cross-efficiency approach calculates average cross-efficiency (ACE) scores, forming the basis for DMU rankings. A higher ACE score signifies superior organizational performance. Nevertheless, the traditional cross-efficiency method faces a fundamental drawback. The optimal weights for inputs and outputs derived from the conventional DEA model lack uniqueness, leading to non-unique cross-efficiency scores. To address this, Doyle and Green [13] introduced an innovative secondary goal to the standard cross-efficiency approach. They proposed aggressive and benevolent models to identify the ideal DEA weights for a DMU. The benevolent (or aggressive) model maintains a DMU's relative efficiency score while optimizing (or reducing) the relative efficiency scores of other DMUs through the best possible DEA weights. However, these models may generate different rankings for similar cases due to their distinct perspectives. Subsequently, various mathematical models emerged based on the concept of secondary goals in cross-efficiency measurement. For instance, Liang *et al.* [14] developed three alternative cross-efficiency models with secondary goals, integrating the notion of optimal spot to address common challenges. Liang *et al.* [15] introduced the game cross-efficiency method, integrating competitive elements, to assess efficiency within the DEA framework. They also explored the concept of Nash equilibrium in this context. This research enhances our understanding of how DMUs can enhance their efficiency while considering the competitive dynamics that influence their operations. Further advancements include Wang and Chin [16] extension of Liang's models [14] by defining the true ideal point and altering the efficiency target. Wang and Chin [17] proposed a neutral cross-efficiency model to combat discrimination among DMUs. Jahanshahloo [18] introduced a method of symmetric weighting to reward DMUs for balanced weighting decisions without compromising feasibility. Additionally, several neutral cross-efficiency models based on ideal and anti-ideal DMUs were proposed [19], [20], and DEA-CE techniques were developed based on weight-balanced models and Pareto optimization [21]. Abolghasem *et al.* [22] incorporated flexible measures into the aggressive and benevolent models for the DEA-CE technique.

Despite these developments, the problem of unique efficiency persists, as the optimal DEA weights derived from a CCR model are frequently not unique. Nevertheless, the game cross-efficiency approach provides a solution by generating unique cross-efficiency values through pairwise games between competing DMUs, while preserving the efficiency of other DMUs. Each DMU is viewed as an individual seeking to maximize its own efficiency, with the assumption that the cross-efficiency of other DMUs remains unaffected. The optimal game cross-efficiency scores are determined by the iterative nature of the game cross-efficiency model, and various initial scores result in identical cross-efficiency outcomes, representing a Nash equilibrium. This method has gained widespread acceptance and application across various domains, including supplier selection [23], [24], urban public infrastructure investment [25], [26], ecological efficiency surveys [27], [28], energy efficiency [29], [30], forest carbon sequestration [31]–[38]. However, it's important to note that the game cross-efficiency technique may not be suitable for ranking all DMUs in certain DEA ranking problems, as indicated by the literature review. In this study, we merge the optimistic and pessimistic aspects of the traditional game cross-efficiency model to address data envelopment analysis ranking problems. After compiling a game interval cross-efficiency decision matrix, Gibbs entropy information is leveraged to rank all DMUs based on interval data. The following section provides a literature review on entropy information.

Entropy formulation is an effective and extensively employed weighting method for evaluating the uncertainty of data. According to the concept of entropy, the integrity of the information is one of the most crucial factors in determining the best course of action. In determining the weights of criteria in DEA ranking problems [39]–[41], the entropy approach of Shannon is frequently employed. Recently, however, the application of entropy to interval DEA ranking problems has been presented and has become a topic of interest. Wang *et al.* [41] initially utilized a DEA entropy model to convert interval values of cross-efficiency into precise relative efficiencies, and all DMUs can be arranged according to the positive ideal distance. In order to rank all DMUs, Lu and Liu [42] proposed a Gibbs entropy optimization model to transform interval

cross-efficiency scores into precise entropy scores. This model is user-friendly and can be calculated using optimization software. Nonetheless, the original Gibbs entropy model is classified as a nonlinear programming model; employing the optimization solver to identify optimal entropy solutions for large problems can be extremely difficult. In order to rank all DMUs in this study, the optimization model based on the original Gibbs entropy model [42] must be converted into a linear programming model. This research presents a hybrid strategy for ranking all DMUs based on an optimistic–pessimistic game cross-efficiency method. The following are the principal contributions of this research:

- Based on the original Gibbs entropy model [42], this model is categorized as a nonlinear programming model. Obtaining optimal solutions with optimization software may be challenging. This study introduces the Gibbs entropy linear programming model, a novel linear programming model based on Gibbs entropy concepts, for ranking DMUs with interval data.
- We apply the proposed method to a real-world scenario that includes twenty provinces in Thailand with data on revenue crops. This will be immensely beneficial for research in this sector in practically every country, especially agricultural nations.

The remainder of this paper is as follows: next, some cross-efficiency models and the original Gibbs entropy model are presented. Section 2 then presents a novel Gibbs entropy linear programming model that combines the optimistic and pessimistic perspectives of the traditional game cross-efficiency method for data envelopment analysis ranking problems. In section 3, verification is conducted for three numerical examples, six nursing institutions, numerical example 2, and twenty Thai provinces. Section 4 concludes with the conclusions.

2. METHOD

This section introduces the Gibbs entropy-based model, an innovative linear programming approach. It combines optimistic and pessimistic viewpoints from the game cross-efficiency model to address ranking problems in DEA. The model accurately determines weights using linear programming techniques, providing decision-makers with a comprehensive assessment of DMU performance. Figure 1 visually depicts the framework of the proposed model.

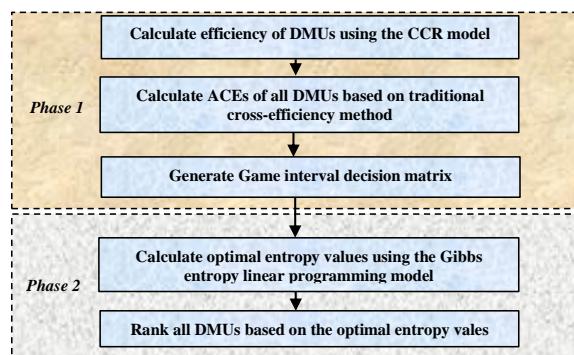


Figure 1. The proposed framework

2.1. CCR model

Charnes *et al.* [2] proposed a mathematical model known as DEA-CCR model, which was designed to evaluate the performance of a collection of DMUs that possess various inputs and outputs. This model assumes that each DMU_j (where $j = 1, 2, 3, \dots, n$) possesses a collection of multi-inputs (x_{ij}), where $i = 1, 2, 3, \dots, m$, and generates a set of multi-outputs (y_{rj}), where $r = 1, 2, \dots, s$. Let v_{ik} , where $k = 1, 2, 3, \dots, n$, be the input weight for each DMU_k . Let u_{rk} , where $k = 1, 2, 3, \dots, n$, be the output weight for each DMU_k . The efficiency score (E_{dd}) for a set of DMU_d ($1 \leq d \leq n$) can be measured using the CCR model, as shown in model (1).

$$\begin{aligned}
 E_{dd} = & \max \sum_{r=1}^s u_{rd} y_{rd} \\
 \text{s. t.: } & \sum_{r=1}^s u_{rd} y_{rj} - \sum_{i=1}^m v_{id} x_{ij} \leq 0, j, d = 1, 2, 3, \dots, n \\
 & \sum_{i=1}^m v_{id} x_{id} = 1, d = 1, 2, 3, \dots, n \\
 & u_{rd} \geq 0, v_{id} \geq 0, i = 1, 2, 3, \dots, m, r = 1, 2, 3, \dots, s
 \end{aligned} \tag{1}$$

In model (1), each DMU constructs the programming and selects the best DEA weights for its inputs and outputs to maximize efficiency, which could contribute to biases in how DMUs are ranked for efficiency when comparing their efficiency scores. This is the reason why the cross-efficiency method has been proposed as a solution to this issue.

2.2. Traditional cross-efficiency method

The cross-efficiency method is a potent and widely adopted traditional CCR model that utilizes self-assessment and peer-assessment to evaluate and rank DMUs with multiple inputs and multiple outputs. The CCR model's calculation stages are as follows. After solving the CCR model in model (1), let and represent the optimal output and input weights for a particular DMU_d , respectively. Then, the cross-efficiencies of each DMU_j ($j = 1, 2, 3, \dots, n$) are provided by DMU_d .

$$E_{dj} = \sum_{r=1}^s u_{rd}^* y_{rj} / \sum_{i=1}^m v_{id}^* x_{ij}, d, j = 1, 2, 3, \dots, n \quad (2)$$

Sexton *et al.* [9] consequently defined the ACE score of DMU_j as (3):

$$\bar{E}_j = (1/n) \sum_{d=1}^n E_{dj}, d, j = 1, 2, \dots, n \quad (3)$$

If the \bar{E}_j (ACE score) of a DMU is higher, it is better ordered. However, the cross-efficiency method may encounter a problem with multiple solutions; consequently, numerous researchers have sought to enhance the traditional cross-efficiency method by incorporating secondary objectives into the traditional model.

2.3. Generating the game interval cross-efficiency matrix

By solving (1), the optimal weights of the inputs and outputs can be determined. The ACE scores of each DMU can then be determined (3) using (2). The optimistic-pessimistic perspectives of the classical game cross-efficiency model are then constructed in order to construct the game interval decision matrix. The details of the optimistic and pessimistic game cross-efficiency models are as (4):

$$\begin{aligned} Z_d = \max \text{ or } \min \sum_{r=1}^s u_{rj}^d y_{rj} \\ \text{s. t.: } \sum_{i=1}^m v_{ij}^d x_{il} - \sum_{r=1}^s u_{rj}^d y_{rl} \geq 0, \\ \sum_{i=1}^m v_{ij}^d x_{ij} = 1, \\ \alpha_d \sum_{i=1}^m v_{ij}^d x_{id} - \sum_{r=1}^s u_{rj}^d y_{rd} \leq 0, \\ v_{ij}^d \geq 0, u_{rj}^d \geq 0, r = 1, 2, 3, \dots, s, i = 1, 2, 3, \dots, m. \end{aligned} \quad (4)$$

To obtain the optimistic and pessimistic scores of DMUs, model (4) is run twice based on the objective function: by solving model (4) with the three steps of the iterative algorithm described in the literature by Liang *et al.* [15], the game interval cross-efficiency matrix based on the optimistic ($\max Z$) and pessimistic ($\min Z$) viewpoints can be generated, as shown in Table 1, where DMU_j ($j = 1, 2, 3, \dots, n$) can be viewed as the alternative j , and iteration $t(I_t)$ can be viewed as the criterion t ($t = 1, 2, 3, \dots, m$). Let α_j^{ut} and α_j^{lt} be the optimistic game cross-efficiency score and the pessimistic game cross-efficiency score, respectively, for DMU_j and I_t .

Table 1. The game interval cross-efficiency matrix

DMU_i	I_1	I_2	\dots	I_m
1	$[\alpha_1^{l1}, \alpha_1^{u1}]$	$[\alpha_1^{l2}, \alpha_1^{u2}]$	\dots	$[\alpha_1^{lm}, \alpha_1^{um}]$
2	$[\alpha_2^{l1}, \alpha_2^{u1}]$	$[\alpha_2^{l2}, \alpha_2^{u2}]$	\dots	$[\alpha_2^{lm}, \alpha_2^{um}]$
\vdots	\vdots	\vdots	\vdots	\vdots
n	$[\alpha_n^{l1}, \alpha_n^{u1}]$	$[\alpha_n^{l2}, \alpha_n^{u2}]$	\dots	$[\alpha_n^{lm}, \alpha_n^{um}]$

2.4. Calculating the optimal entropy values using the novel Gibbs entropy linear programming model

Given that the original Gibbs entropy model [42] is classified as a non-linear programming model, it may be challenging to obtain optimal solutions using optimization software. Therefore, it is necessary to modify the original model to integrate linear programming. This study presents an original Gibbs entropy linear programming model for ranking all DMUs based on the optimistic and pessimistic perspectives of the

game cross-efficiency model. The following details pertain to the proposed Gibbs entropy linear programming model. Using the original Gibbs entropy model [42], models (5) through (8) illustrate how the proposed model could be derived.

$$\begin{aligned} \bar{H}_j &= \min \left[-G_j \sum_{t=1}^m \left(\bar{\alpha}_j^t / \sum_{t=1}^m \bar{\alpha}_j^t \right) \ln \left(\bar{\alpha}_j^t / \sum_{t=1}^m \bar{\alpha}_j^t \right) \right], \forall j, \\ \text{s.t.:} & \sum_{t=1}^m \left(\bar{\alpha}_j^t / \sum_{t=1}^m \bar{\alpha}_j^t \right) = 1, \forall j, \\ \alpha_{ij}^{lt} &\leq \bar{\alpha}_j^t \leq \alpha_{ij}^{ut}, \forall t, \forall j. \end{aligned} \quad (5)$$

$$\begin{aligned} \bar{H}_j &= \min \left[-G_j \sum_{t=1}^m \left(\left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) / \sum_{j=1}^n \left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) \right) \ln \left(\left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) / \sum_{j=1}^n \left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) \right) \right], \forall j, \quad (6) \\ \text{s.t.:} & \sum_{t=1}^m \left(\left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) / \sum_{j=1}^n \left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) \right) = 1, \forall j, \end{aligned}$$

where G_j is the cross-efficiency score for the final game for DMU_j (constant value). Using division, model (6) can be transformed into model (7).

$$\begin{aligned} \bar{H}_j &= \min \left[-G_j \sum_{t=1}^m \left(\left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) / \sum_{j=1}^n \left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) \right) \ln \left(\left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) / \sum_{j=1}^n \left((\alpha_j^{lt} + \alpha_j^{ut})/2 \right) \right) \right], \forall j, \quad (7) \\ \text{s.t.:} & \sum_{t=1}^m \left(\left(\alpha_j^{lt} / \sum_{t=1}^m \left(\alpha_j^{lt} + \alpha_j^{ut} \right) \right) + \left(\alpha_j^{ut} / \sum_{t=1}^m \left(\alpha_j^{lt} + \alpha_j^{ut} \right) \right) \right) = 1, \forall j, \end{aligned}$$

Model (7) is a nonlinear programming model. Set t_j as $t_j = 1 / \sum_{t=1}^m (\alpha_j^{lt} + \alpha_j^{ut})$. This model can be converted to a linear programming model as shown in model (8).

$$\begin{aligned} \bar{H}_j &= \min \left(-G_j \sum_{t=1}^m \left((\alpha_j^{lt} + \alpha_j^{ut}) t_j \ln (\alpha_j^{lt} + \alpha_j^{ut}) t_j \right) \right), \forall j, \\ \text{s.t.:} & \sum_{t=1}^m \left((\alpha_j^{lt} + \alpha_j^{ut}) t_j \right) = 1, \forall j, \\ t_j &\geq 0, \forall j \end{aligned} \quad (8)$$

where G_j is the constant value of DMU_j and t_j is the decision variable for DMU_j . \bar{H}_j^* is the optimal entropy value. If the optimal entropy value (\bar{H}_j^*) of a DMU is higher, it is better ordered.

3. RESULTS

In this section, the research outcomes are described alongside a thorough analysis. Results can be presented in figures, graphs, and tables that facilitate reader comprehension [14], [15]. The discussion may be divided into multiple subsections.

3.1. The six nursing homes problem

The six-nursing home problem was presented by Sexton *et al.* [9] with two inputs and two outputs. Let x_1, x_2, y_1 , and y_2 represent the staff hours per day, the supplies per day, the total medicare-plus-medicaid patient days, and the total privately paid patient days, respectively. Table 2 displays the data set for the six nursing homes problem.

Table 2. Set of data for the six nursing homes problem

DMU_j	x_1	x_2	y_1	y_2	CCR
1	1.50	0.20	1.40	0.35	1.0000
2	4.00	0.70	1.40	2.10	1.0000
3	3.20	1.20	4.20	1.05	1.0000
4	5.20	2.00	2.80	4.20	1.0000
5	3.50	1.20	1.90	2.50	0.9775
6	3.20	0.70	1.40	1.50	0.8675

3.1.1. Generating the game interval cross-efficiency matrix based on the optimistic and pessimistic viewpoints for the six nursing homes

Based on the data set for the six nursing homes regarding the inputs and outputs of each DMU_j listed in Table 2, the CCR scores based on model (1) were coded using LINGO software. After obtaining the optimal weights for the inputs and outputs, the ACE score of each DMU (\bar{E}_j) was obtained using (2) to (3). As a result, the values of $\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4, \bar{E}_5$, and \bar{E}_6 were calculated to be 0.8529, 0.8259, 0.7643, 0.8510,

0.8316, and 0.7286 respectively. In this research, the cross-efficiency score of the arbitrary strategy was set with an initial value of α_j^1 , ε set as 0.001. For iteration 1 (criterion 1 or I_1), for each DMU_j, if $\alpha_j^1 = \bar{E}_d$ ($j = d$), then $\alpha_1^1=0.8529$, $\alpha_2^1=0.8259$, $\alpha_3^1=0.7643$, $\alpha_4^1=0.8510$, $\alpha_5^1=0.8316$, and $\alpha_6^1=0.7286$. These parameters were taken into model (4) to generate the game interval cross-efficiency scores for iteration 2. For iteration 2, using model (4), through 3 steps of the iterative algorithm, the optimistic cross-efficiency scores (max Z_d) of iterations 2 ($t = 2$) for each DMU_j were determined to be $\alpha_1^1=1.0000$, $\alpha_2^1=1.0000$, $\alpha_3^1=1.0000$, $\alpha_4^1=1.0000$, $\alpha_5^1=0.9775$, and $\alpha_6^1=0.8675$, respectively. The pessimistic cross-efficiency scores (min Z_d) of iterations 2 ($t = 2$) for each DMU_j were obtained as $\alpha_1^1=0.4532$, $\alpha_2^1=0.5276$, $\alpha_3^1=0.4068$, $\alpha_4^1=0.5689$, $\alpha_5^1=0.5580$, and $\alpha_6^1=0.4761$. As a result, the game interval cross-efficiency matrix shown in Table 3 was generated. The attainment of scores for optimistic game cross-efficiency for all DMUs during the seventh iteration (I_7) is demonstrated in Table 3. The cross-efficiency scores for DMU1, DMU2, DMU3, DMU4, DMU5, and DMU6 in the final game were 1.00, 0.9868, 0.9221, 1.00, 0.9766, and 0.8615, respectively (shown in bold).

Table 3. The game interval cross-efficiency matrix for the six nursing homes

DMU _j	I_1	I_2	I_3	I_4	I_5	I_6	I_7
1	[0.8529, 0.8529] [0.4532, 1.0000]	[0.7496, 1.0000] [0.7184, 1.0000]	[0.7300, 1.0000] [0.7277, 1.0000]	[0.7286, 1.0000] [0.7286, 1.0000]			
2	[0.8259, 0.8259] [0.5276, 1.0000]	[0.7004, 0.9773] [0.6847, 0.9879]	[0.6910, 0.9861] [0.6898, 0.9870]	[0.6903, 0.9868] [0.6903, 0.9868]			
3	[0.7643, 0.7643] [0.4068, 1.0000]	[0.6428, 0.9148] [0.6120, 0.9280]	[0.6222, 0.9216] [0.6199, 0.9226]	[0.6206, 0.9221] [0.6206, 0.9221]			
4	[0.8510, 0.8510] [0.5689, 1.0000]	[0.7176, 0.9176] [0.7030, 0.9708]	[0.7080, 0.9765] [0.7068, 0.9766]	[0.7072, 0.9766] [0.7072, 0.9766]			
5	[0.8316, 0.8316] [0.5580, 0.9775]	[0.6956, 0.9758] [0.6818, 0.9767]	[0.6864, 0.9765] [0.6854, 0.9766]	[0.6858, 0.9766] [0.6858, 0.9766]			
6	[0.7286, 0.7286] [0.4761, 0.8675]	[0.6081, 0.8570] [0.5971, 0.8620]	[0.6008, 0.8611] [0.5999, 0.8615]	[0.6002, 0.8615] [0.6002, 0.8615]			

3.1.2. Rank all DMUs using the Gibbs entropy linear programming model for the six nursing homes

The game interval cross-efficiency matrix Table 3 was obtained and afterwards, the suggested Gibbs entropy model was employed to convert the interval cross-efficiency scores into crisp scores. This conversion was necessary in order to rank the DMUs comprehensively. In order to acquire the values of \bar{H}_j^* , the pertinent parameters enumerated in Table 3 were inputted into model (8). An illustrative instance of a linear programming model is employed to determine the best value of entropy, denoted as \bar{H}_j^* .

$$\bar{H}_1 = \min \left(-1.0000 \left(\begin{array}{l} ((0.8529 + 0.8529)t_1 \ln(0.8529 + 0.8529) t_1 + (0.4532 + 1.0000)t_1 \ln(0.4532 + 1.0000) t_1 + (0.7496 + 1.0000)t_1 \ln(0.7496 + 1.0000) t_1) \\ \ln(0.7496 + 1.0000) t_1 + (0.7184 + 1.0000)t_1 \ln(0.7184 + 1.0000) t_1 + (0.7300 + 1.0000)t_1 \ln(0.7300 + 1.0000) t_1 \\ + (0.7277 + 1.0000)t_1 \ln(0.7277 + 1.0000) t_1 + (0.7286 + 1.0000)t_1 \ln(0.7286 + 1.0000) t_1 \end{array} \right) \right)$$

$$\text{s.t.: } (0.8529 + 0.8529)t_1 + (0.4532 + 1.0000)t_1 + (0.7496 + 1.0000)t_1 \\ + (0.7184 + 1.0000)t_1 + (0.7300 + 1.0000)t_1 + (0.7277 + 1.0000)t_1 \\ + (0.7286 + 1.0000)t_1 = 1,$$

$$t_1 \geq 0.$$

To determine the optimal value of \bar{H}_1^* , this LP model for \bar{H}_1^* was solved using the LINGO software. The optimal value of \bar{H}_1^* was determined to be 1.9442, occurring at $t_1^*=0.08465$. With the same calculation steps, the other values of \bar{H}_j^* ($j=2, 3, \dots, 6$) were determined to be 1.9198, 1.7939, 1.9455, 1.9000, and 1.6760, respectively. The other values of t_j^* ($j=2, 3, \dots, 6$) were determined to be 0.08650, 0.09379, 0.08465, 0.08684, and 0.09891, respectively. Based on the obtained optimal entropy values, all the DMUs could be fully ranked. The ranking comparisons between the proposed technique and other methods are presented in Table 4 for all the DMUs.

The rankings for all the DMUs were computed using the suggested technique, as indicated in Table 4. The suggested technique has a tendency towards consistency with previous cross-efficiency methods. Furthermore, it is important to acknowledge that the original game cross-efficiency algorithm lacks the ability to distinguish between DMU₁ and DMU₄. The results of the suggested technique are compared with those of the original game cross-efficiency method, as seen in Figure 2. The graphic shown in this analysis demonstrates a notable alignment between the suggested method and the original game cross-efficiency approach. The two methodologies yielded differing rankings for just the DMU₁.

Table 4. Ranking comparisons between the proposed method and the alternative methods for the six nursing homes problem

DMU _j	Aggressive (rank)	Benevolent (rank)	Game (rank)	Proposed (rank)	Original model (rank)
1	0.7639 (1)	1.0000 (1)	1.0000 (1)	1.9442 (2)	1.9173 (2)
2	0.7004 (3)	0.9773 (3)	0.9868 (3)	1.9198 (3)	1.8982 (3)
3	0.6428 (5)	0.8580 (5)	0.9221 (5)	1.7939 (5)	1.7642 (5)
4	0.7184 (2)	1.0000 (1)	1.0000 (1)	1.9455 (1)	1.9261 (1)
5	0.6956 (4)	0.9758 (4)	0.9766 (4)	1.9000 (4)	1.8814 (4)
6	0.6081 (6)	0.8570 (6)	0.8615 (6)	1.6760 (6)	1.6579 (6)

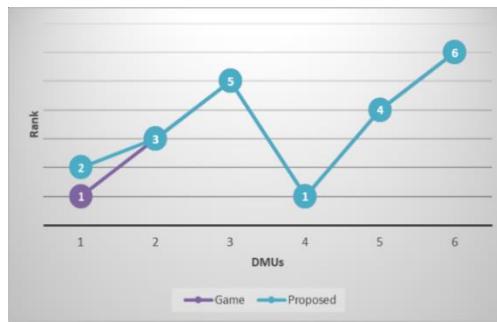


Figure 2. The ranking comparisons for the six nursing homes

Furthermore, the statistical analysis included the examination of Spearman's correlation coefficient (r_s). Consequently, the r_s for the suggested approach, namely the aggressive, benevolent, and game cross-efficiency methods, were calculated as 0.939, 0.900, and 0.998 correspondingly. It is noteworthy to mention that the suggested ranking methodology exhibits a significant association with the widely recognized cross-efficiency approaches. Besides, the proposed Gibbs entropy linear programming model was compared with the Gibbs entropy optimization model of Lu and Liu [42] for solving this problem. The results show that the ranks of each DMU were the same for both models.

3.2. The numerical example 2

In numerical example 2 provided by Liang *et al.* [15], there were ten DMUs with two inputs (x_1 and x_2) and three outputs (y_1 , y_2 and y_3) each. The CCR scores for each DMU were calculated using the traditional CCR model, followed by the proposed game cross-efficiency method for calculating the interval cross-efficiency scores for each DMU. Table 5 displays the information for numerical example 2.

Table 5. The numerical example 2's dataset

DMU _j	x_1	x_2	y_1	y_2	y_3	CCR
1	0.37589	0.19389	0.62731	0.71654	0.11461	1.0000
2	0.00988	0.90481	0.69908	0.51131	0.66486	1.0000
3	0.41986	0.56921	0.39718	0.77640	0.36537	0.7590
4	0.75367	0.63179	0.41363	0.48935	0.14004	0.3099
5	0.79387	0.23441	0.65521	0.18590	0.56677	1.0000
6	0.91996	0.54878	0.83759	0.70064	0.82301	0.7155
7	0.84472	0.93158	0.37161	0.98271	0.67395	0.5062
8	0.36775	0.33520	0.42525	0.80664	0.99945	1.0000
9	0.62080	0.65553	0.59466	0.70357	0.96164	0.6608
10	0.73128	0.39190	0.56574	0.48496	0.05886	0.4594

3.2.1. Generating the game interval cross-efficiency matrix for numerical example 2 based on optimistic and pessimistic perspectives

Based on the inputs and outputs of each DMU_j listed in Table 6, the CCR scores and the values of each \bar{E}_j were calculated using the same calculation steps as shown in subsubsection 3.1.1. As a result, the cross-efficiency score of the arbitrary model was set at an initial value of α_j^1 , ε set as 0.001. Finally, Table 6 displays the results of the game interval cross-efficiency matrix.

3.2.2. Rank all DMUs using the Gibbs entropy linear programming model for numerical example 2

After obtaining the game interval cross-efficiency matrix for numerical example 2 and using the same calculation procedures of the proposed Gibbs entropy linear programming model as presented in *An optimistic-pessimistic game cross-efficiency method based on a Gibbs entropy ... (Noppakun Thongmual)*

subsubsection 3.1.2. the proposed Gibbs entropy linear programming model was utilized to rank the DMUs exhaustively. The ranking comparisons between the proposed method and the other methods for each DMU are displayed in Table 7. The rankings for all the DMUs were computed using the proposed method, as shown in Table 7. The suggested technique has a tendency towards consistency with previous cross-efficiency methods. Furthermore, it is important to acknowledge that the game cross-efficiency approaches lack the ability to distinguish between DMU₁ and DMU₈. Figure 3 illustrates the outcomes obtained from the suggested methodology in comparison with the conventional game cross-efficiency approach. The graphic illustrates a strong association between the suggested approach and the game cross-efficiency method. The two techniques yielded different rankings for just DMU₁.

Table 6. The game interval cross-efficiency matrix for the numerical example 2

DMU _j	I ₁	I ₂	I ₃	I ₄	I ₅	I ₆	I ₇	I ₈	I ₉	I ₁₀	I ₁₁	I ₁₂ *
1	[0.9492, 0.9492]	[0.4270, 1.0000]	[0.6900, 1.0000]	[0.5748, 1.0000]	[0.6324, 1.0000]	[0.6071, 1.0000]	[0.6192, 1.0000]	[0.6136, 1.0000]	[0.6162, 1.0000]	[0.6150, 1.0000]	[0.6155, 1.0000]	[0.6153, 1.0000]
2	[0.8887, 0.8887]	[0.3095, 1.0000]	[0.5361, 0.9395]	[0.4091, 1.0000]	[0.4645, 0.9732]	[0.4380, 0.9850]	[0.4496, 0.9794]	[0.4441, 0.9819]	[0.4466, 0.9807]	[0.4455, 0.9812]	[0.4460, 0.9810]	[0.4458, 0.9811]
3	[0.5650, 0.5650]	[0.2560, 0.7208]	[0.3814, 0.6364]	[0.3260, 0.6673]	[0.3526, 0.6506]	[0.3404, 0.6577]	[0.3460, 0.6543]	[0.3434, 0.6559]	[0.3446, 0.6552]	[0.3441, 0.6555]	[0.3443, 0.6554]	[0.3442, 0.6554]
4	[0.2775, 0.2775]	[0.1256, 0.3077]	[0.1925, 0.2989]	[0.1626, 0.3043]	[0.1771, 0.3018]	[0.1707, 0.3030]	[0.1737, 0.3025]	[0.1724, 0.3027]	[0.1730, 0.3026]	[0.1727, 0.3026]	[0.1728, 0.3026]	[0.1728, 0.3026]
5	[0.4398, 0.4398]	[0.1741, 0.8517]	[0.3434, 0.6853]	[0.2774, 0.7807]	[0.3129, 0.7403]	[0.2973, 0.7575]	[0.3046, 0.7494]	[0.3012, 0.7532]	[0.3028, 0.7514]	[0.3021, 0.7522]	[0.3024, 0.7519]	[0.3023, 0.7520]
6	[0.4960, 0.4960]	[0.3005, 0.6805]	[0.4087, 0.6129]	[0.3657, 0.6522]	[0.3887, 0.6347]	[0.3787, 0.6437]	[0.3834, 0.6396]	[0.3812, 0.6415]	[0.3822, 0.6406]	[0.3818, 0.6410]	[0.3820, 0.6408]	[0.3819, 0.6409]
7	[0.3716, 0.3716]	[0.1794, 0.4791]	[0.2608, 0.4257]	[0.2260, 0.4440]	[0.2432, 0.4346]	[0.2353, 0.4388]	[0.2390, 0.4368]	[0.2373, 0.4377]	[0.2381, 0.4373]	[0.2377, 0.4375]	[0.2379, 0.4374]	[0.2378, 0.4374]
8	[0.9056, 0.9056]	[0.4992, 1.0000]	[0.6944, 0.9903]	[0.6159, 1.0000]	[0.6559, 1.0000]	[0.6382, 1.0000]	[0.6465, 1.0000]	[0.6426, 1.0000]	[0.6444, 1.0000]	[0.6436, 1.0000]	[0.6440, 1.0000]	[0.6438, 1.0000]
9	[0.5207, 0.5207]	[0.3062, 0.6511]	[0.3979, 0.6103]	[0.3610, 0.6373]	[0.3789, 0.6272]	[0.3710, 0.6321]	[0.3747, 0.6299]	[0.3730, 0.6309]	[0.3738, 0.6304]	[0.3734, 0.6306]	[0.3736, 0.6305]	[0.3735, 0.6306]
10	[0.3765, 0.3765]	[0.1494, 0.4487]	[0.2617, 0.4248]	[0.2118, 0.4361]	[0.2371, 0.4305]	[0.2260, 0.4331]	[0.2313, 0.4319]	[0.2289, 0.4325]	[0.2300, 0.4322]	[0.2295, 0.4323]	[0.2297, 0.4323]	[0.2296, 0.4323]

Table 7. Comparisons between the proposed Gibbs entropy model and the other methods for numerical example 2

DMU _j	Aggressive (rank)	Benevolent (rank)	Game (rank)	Proposed (rank)	Original model (rank)
1	0.8150 (1)	0.9928 (1)	1.0000 (1)	2.4830 (2)	2.4549 (2)
2	0.7153 (3)	0.8945 (3)	0.9811 (3)	2.4354 (3)	2.3587 (3)
3	0.4382 (7)	0.5344 (7)	0.6554 (5)	1.6282 (5)	1.5943 (5)
4	0.2240 (10)	0.2862 (10)	0.3026 (10)	0.7515 (10)	0.7412 (10)
5	0.4705 (4)	0.5786 (6)	0.7520 (4)	1.8679 (4)	1.7884 (4)
6	0.4486 (5)	0.5792 (5)	0.6409 (6)	1.5925 (6)	1.5727 (6)
7	0.3014 (9)	0.3523 (9)	0.4374 (8)	1.0867 (8)	1.0661 (8)
8	0.7843 (2)	0.9587 (2)	1.0000 (1)	2.4841 (1)	2.4628 (1)
9	0.4394 (6)	0.5802 (4)	0.6306 (7)	1.5669 (7)	1.5508 (7)
10	0.3190 (8)	0.4122 (8)	0.4323 (9)	1.0737 (9)	1.0536 (9)

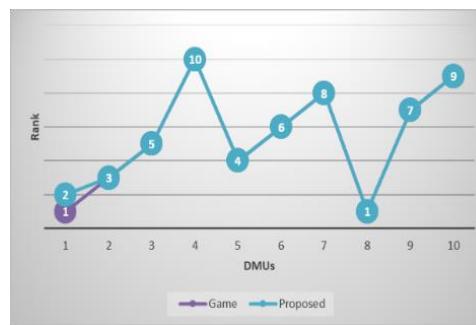


Figure 3. The comparisons of ranks for the numerical example 2

Additionally, the r_s was evaluated. Consequently, $r_s=0.939$, 0.900, and 0.998 were calculated for the proposed method, the aggressive, beneficent, and game cross-efficiency methods, respectively. Notably, the

correlation between the proposed ranking method and well-known cross-efficiency methods is quite high. Besides, the proposed Gibbs entropy linear programming model was compared with the Gibbs entropy optimization model of [42] for solving this problem. The results show that the ranks of each DMU were the same for both models.

3.3. The twenty Thai provinces application

In Thailand, agriculture plays an important role in rural life, trade incomes, food security, and domestic economic development. Agriculture is also the backbone of food industries because it meets their demand for raw materials. Therefore, one of the main national goals of the Thai government is to increase the agricultural productivity of cash crops. The government has promoted a new economic model in a systematic manner with the aim of achieving the Government's Thailand 4.0 vision by focusing on 10 targeted S-curve industries; one of them is the agricultural sector. In the Northeastern region of Thailand, agriculture remains the largest sector, and rice, maize, cassava, sugar cane, and palm are the main cash crops. Agricultural productivity in the poorest regions remains less efficient compared with other regions due to the inefficient use of inputs. This results in reduced efficiency and weak planning, which often leads to inefficient policy making in agriculture, in terms of budget allocation, technology, infrastructure, and other resources. Agricultural productivity can be viewed as the ratio of agricultural outputs to inputs. In some cases, there are multiple production units with multiple inputs and outputs, and input and output measurements have different units. It is exceedingly difficult to calculate agricultural productivity since this problem is complex. Unquestionably, measuring efficiency and ranking the provinces based on their use of these input factors are crucial for establishing appropriate government policies for the economic growth of each province. Planning and formulating policies and related actions to further develop the nation's economy would be greatly aided by the discovery of a dependable instrument for measuring the efficacy and classification of each province. This problem has twenty DMUs, including three inputs (x_1 , x_2 and x_3) and five outputs (y_1 , y_2 , y_3 , y_4 and y_5). Inputs: x_1 , x_2 and x_3 are the number of farmers (persons), the provincial minimum wage (baht), and the planted area (km^2), respectively. Outputs: y_1 , y_2 , y_3 , y_4 and y_5 are the production volume of rice (tons), the production volume of maize (tons), the production volume of cassava (tons), the production volume of sugarcane (tons), and the production volume of oil palm (tons), respectively. DMUs: The twenty DMUs are Loei (DMU₁), Nong Bua Lamphu (DMU₂), Udon Thani (DMU₃), Nong Khai (DMU₄), Bueng Kan (DMU₅), Sakon Nakhon (DMU₆), Nakhon Phanom (DMU₇), Mukdahan (DMU₈), Kalasin (DMU₉), Khon Kaen (DMU₁₀), Maha Sarakham (DMU₁₁), Roi Et (DMU₁₂), Nakhon Ratchasima (DMU₁₃), Chaiyaphum (DMU₁₄), Buriram (DMU₁₅), Surin (DMU₁₆), Yasothon (DMU₁₇), Sisaket (DMU₁₈), Amnat Charoen (DMU₁₉), and Ubon Ratchathani (DMU₂₀), details of this problem are shown in Table 8.

Table 8. The data set for the twenty Thai provinces

DMU _j	x_1	x_2	x_3	y_1	y_2	y_3	y_4	y_5	CCR
1	129638	320	2305	158423	258194	1025676	2725505	39280	1.0000
2	126715	315	1843	262373	18303	337508	2872516	6302	1.0000
3	315405	320	5046	765891	1845	1599257	5936670	35855	1.0000
4	87995	325	1280	270757	147	73077	613127	34060	1.0000
5	55107	320	882	160015	0	21676	35613	50666	1.0000
6	283207	323	4119	793565	62	517047	859419	26955	0.9774
7	177004	320	2575	560177	0	80345	128048	8424	1.0000
8	86845	323	1263	188016	0	495264	1927371	4813	1.0000
9	269444	323	3919	736558	113	370968	3836298	7900	1.0000
10	330834	325	5293	786188	3120	870198	5219604	2267	0.9457
11	256394	315	4102	849444	0	554556	1100761	128	1.0000
12	389831	320	5670	1139001	0	245245	1432720	2728	1.0000
13	664858	325	10638	1276037	468945	4908505	4337480	12437	1.0000
14	361593	315	5260	547606	52725	1920715	3799033	7350	0.8958
15	386095	320	5616	1014534	365	1462651	977755	10539	0.9907
16	375448	320	5461	1138049	46	561824	928355	6516	1.0000
17	177939	320	2588	512585	0	370968	856559	3834	0.9717
18	369434	315	5374	1053588	21624	773525	81534	10786	0.9755
19	142038	315	2066	366439	273	459639	841473	8388	0.9486
20	435969	325	7751	1481730	53465	1818897	28968	30944	1.0000

3.3.1. Generating the game interval cross-efficiency matrix for the twenty Thai provinces based on optimistic and pessimistic perspectives

Based on the dataset provided in Table 8, which contains information on the inputs and outputs of each DMU_j from the twenty Thai provinces, there were a total of twenty DMUs. Each DMU was characterized by two inputs (x_1 , x_2 , and x_3) and three outputs (y_1 , y_2 , y_3 , y_4 , and y_5). To evaluate the performance of each DMU, the CCR and ACE scores were computed using (1) to (3). The interval cross-

efficiency scores for each DMU were then calculated using the proposed game cross-efficiency method. The pertinent information for this instance can be found in Table 9.

Table 9. The interval cross-efficacy matrix for the twenty provinces in Thailand

DMU _j	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}	I_{11}	I_{12}	I_{13}	I_{14}^*
1	[0.895, 0.895] [0.291, 1.000]	[0.497, 0.957]	[0.370, 1.000]	[0.462, 0.993]	[0.407, 1.000]	[0.438, 0.996]	[0.419, 1.000]	[0.431, 0.997]	[0.423, 0.999]	[0.428, 0.998]	[0.425, 0.998]	[0.427, 0.998]	[0.427, 0.998]	[0.426, 0.998]
2	[0.774, 0.774] [0.339, 0.993]	[0.526, 0.899]	[0.433, 0.969]	[0.501, 0.931]	[0.464, 0.956]	[0.487, 0.941]	[0.473, 0.950]	[0.482, 0.944]	[0.476, 0.948]	[0.480, 0.945]	[0.477, 0.947]	[0.479, 0.946]	[0.478, 0.947]	[0.479, 0.947]
3	[0.970, 0.970] [0.558, 1.000]	[0.709, 0.981]	[0.630, 1.000]	[0.683, 0.987]	[0.651, 0.998]	[0.670, 0.991]	[0.658, 0.995]	[0.665, 0.993]	[0.660, 0.994]	[0.663, 0.993]	[0.661, 0.994]	[0.662, 0.993]	[0.662, 0.994]	[0.662, 0.994]
4	[0.856, 0.856] [0.306, 1.000]	[0.508, 0.940]	[0.410, 1.000]	[0.485, 0.985]	[0.446, 1.000]	[0.473, 0.985]	[0.457, 1.000]	[0.467, 1.000]	[0.461, 1.000]	[0.465, 1.000]	[0.462, 1.000]	[0.464, 1.000]	[0.463, 1.000]	[0.463, 1.000]
5	[0.759, 0.759] [0.206, 1.000]	[0.402, 0.891]	[0.296, 0.999]	[0.377, 0.994]	[0.336, 0.964]	[0.365, 0.994]	[0.349, 0.985]	[0.359, 0.993]	[0.353, 0.990]	[0.357, 0.992]	[0.354, 0.991]	[0.356, 0.992]	[0.355, 0.992]	[0.355, 0.992]
6	[0.855, 0.855] [0.422, 0.974]	[0.565, 0.929]	[0.504, 0.964]	[0.550, 0.947]	[0.526, 0.959]	[0.542, 0.952]	[0.533, 0.956]	[0.539, 0.954]	[0.535, 0.956]	[0.538, 0.954]	[0.536, 0.955]	[0.537, 0.955]	[0.537, 0.955]	[0.537, 0.955]
7	[0.807, 0.807] [0.316, 0.997]	[0.501, 0.934]	[0.419, 0.990]	[0.483, 0.967]	[0.451, 0.985]	[0.473, 0.976]	[0.460, 0.982]	[0.469, 0.979]	[0.463, 0.981]	[0.467, 0.979]	[0.465, 0.980]	[0.466, 0.980]	[0.465, 0.980]	[0.465, 0.980]
8	[0.846, 0.846] [0.302, 1.000]	[0.520, 0.928]	[0.409, 0.998]	[0.492, 0.963]	[0.447, 0.991]	[0.476, 0.975]	[0.459, 0.985]	[0.470, 0.978]	[0.463, 0.982]	[0.467, 0.979]	[0.464, 0.981]	[0.466, 0.980]	[0.465, 0.981]	[0.465, 0.981]
9	[0.873, 0.873] [0.463, 1.000]	[0.629, 0.971]	[0.551, 0.999]	[0.607, 0.987]	[0.574, 0.998]	[0.595, 0.991]	[0.584, 0.996]	[0.591, 0.992]	[0.586, 0.995]	[0.589, 0.993]	[0.587, 0.994]	[0.588, 0.993]	[0.588, 0.994]	[0.588, 0.994]
10	[0.791, 0.791] [0.441, 0.942]	[0.569, 0.891]	[0.507, 0.924]	[0.550, 0.902]	[0.524, 0.916]	[0.539, 0.906]	[0.531, 0.912]	[0.536, 0.908]	[0.533, 0.911]	[0.535, 0.909]	[0.533, 0.910]	[0.534, 0.909]	[0.534, 0.910]	[0.534, 0.910]
11	[0.885, 0.885] [0.413, 1.000]	[0.582, 0.981]	[0.511, 1.000]	[0.566, 0.997]	[0.538, 1.000]	[0.556, 0.999]	[0.545, 1.000]	[0.552, 0.999]	[0.548, 1.000]	[0.551, 0.999]	[0.549, 1.000]	[0.550, 0.999]	[0.549, 0.999]	[0.549, 0.999]
12	[0.824, 0.824] [0.397, 0.924]	[0.567, 0.948]	[0.495, 0.982]	[0.551, 0.965]	[0.521, 0.977]	[0.542, 0.969]	[0.530, 0.974]	[0.538, 0.971]	[0.533, 0.973]	[0.536, 0.972]	[0.534, 0.973]	[0.535, 0.972]	[0.535, 0.972]	[0.535, 0.972]
13	[0.936, 0.936] [0.445, 1.000]	[0.608, 0.968]	[0.514, 1.000]	[0.579, 0.998]	[0.535, 0.995]	[0.565, 0.995]	[0.547, 0.995]	[0.558, 0.995]	[0.551, 0.995]	[0.555, 0.995]	[0.552, 0.995]	[0.554, 0.995]	[0.553, 0.995]	[0.553, 0.995]
14	[0.736, 0.736] [0.376, 0.863]	[0.493, 0.794]	[0.430, 0.837]	[0.473, 0.803]	[0.448, 0.823]	[0.463, 0.809]	[0.454, 0.817]	[0.459, 0.812]	[0.456, 0.815]	[0.458, 0.813]	[0.457, 0.814]	[0.457, 0.814]	[0.457, 0.814]	[0.457, 0.814]
15	[0.884, 0.884] [0.422, 0.990]	[0.572, 0.950]	[0.505, 0.983]	[0.555, 0.965]	[0.528, 0.977]	[0.546, 0.969]	[0.535, 0.974]	[0.542, 0.971]	[0.538, 0.973]	[0.541, 0.971]	[0.539, 0.972]	[0.540, 0.972]	[0.539, 0.972]	[0.539, 0.972]
16	[0.883, 0.883] [0.433, 1.000]	[0.596, 0.975]	[0.527, 0.999]	[0.581, 0.990]	[0.553, 0.997]	[0.572, 0.993]	[0.561, 0.996]	[0.568, 0.994]	[0.564, 0.996]	[0.567, 0.995]	[0.565, 0.995]	[0.566, 0.995]	[0.566, 0.995]	[0.566, 0.995]
17	[0.825, 0.825] [0.355, 0.972]	[0.527, 0.924]	[0.453, 0.968]	[0.510, 0.950]	[0.480, 0.964]	[0.500, 0.958]	[0.489, 0.962]	[0.496, 0.959]	[0.491, 0.961]	[0.495, 0.960]	[0.493, 0.960]	[0.494, 0.960]	[0.493, 0.960]	[0.493, 0.960]
18	[0.856, 0.856] [0.388, 0.973]	[0.548, 0.939]	[0.479, 0.969]	[0.532, 0.956]	[0.504, 0.965]	[0.523, 0.959]	[0.512, 0.964]	[0.519, 0.961]	[0.515, 0.963]	[0.518, 0.962]	[0.516, 0.962]	[0.517, 0.962]	[0.516, 0.962]	[0.516, 0.962]
19	[0.799, 0.799] [0.333, 0.948]	[0.496, 0.887]	[0.420, 0.939]	[0.477, 0.915]	[0.447, 0.932]	[0.467, 0.923]	[0.455, 0.929]	[0.463, 0.925]	[0.458, 0.928]	[0.461, 0.926]	[0.459, 0.927]	[0.461, 0.927]	[0.460, 0.927]	[0.460, 0.927]
20	[0.960, 0.960] [0.431, 1.000]	[0.612, 0.989]	[0.531, 1.000]	[0.594, 0.999]	[0.560, 1.000]	[0.583, 1.000]	[0.569, 1.000]	[0.578, 1.000]	[0.572, 1.000]	[0.576, 1.000]	[0.573, 1.000]	[0.575, 1.000]	[0.574, 1.000]	[0.574, 1.000]

3.3.2. Rank all DMUs using the Gibbs entropy linear programming model for numerical example 2

Utilizing the identical computational procedures outlined in subsubsection 3.1.2 for the Gibbs entropy linear programming model, the study applied this model to rank DMUs across twenty Thai provinces, utilizing the game interval cross-efficiency matrix. As delineated in Table 9, the interval cross-efficiency values, initially varying, uniformly converged to a stable game interval cross-efficiency value for each DMU after 14 iterations. This convergence signifies the solution's nature as a Nash equilibrium, a fact substantiated by prior research [15]. Table 10 presents the comparative rankings of the proposed method against alternative cross-efficiency approaches, revealing the proposed method's superior ability to differentiate between DMUs, a shortcoming observed in the game cross-efficiency method, notably concerning DMU₁ and DMU₂₀. Furthermore, the study juxtaposed the proposed Gibbs entropy model with Lu and Liu's Gibbs entropy optimization model [42], yielding a correlation coefficient of $r_s=0.877$. This outcome underscores the model's efficacy in addressing the problem at hand.

As seen in Table 9, an initial interval cross efficiency value for each DMU converges to a stable game interval cross efficiency value, indicating that the solution is a Nash equilibrium, as demonstrated in [15]. Table 9 demonstrates that after 14 iterations, all DMUs calculated by the proposed algorithm attain a constant value for game-cross efficiency. The concurrence between the proposed method and the original game cross-efficiency method is vividly demonstrated in Figure 4. Notably, only DMU₂₀ exhibited divergent rankings between the two methods. Spearman's correlation coefficient (r_s) attested to this strong concordance ($r_s=0.986$), underscoring the effectiveness of the proposed ranking method and its affinity with the game cross-efficiency method.

Table 10. The ranking comparisons of the proposed method and the other cross-efficiency methods for the twenty Thai provinces

DMU _j	Aggressive (rank)	Benevolent (rank)	Game (rank)	Proposed (rank)	Original (rank)
1	0.5993 (11)	0.9034 (8)	0.9983 (4)	2.6321 (4)	2.5357(8)
2	0.6168 (8)	0.8305 (18)	0.9467 (17)	2.4980 (17)	2.4388(17)
3	0.7464 (1)	0.9764 (1)	0.9937 (7)	2.6213 (7)	2.5991(1)
4	0.5616 (15)	0.8657 (14)	1.0000 (1)	2.6376 (1)	2.5575(7)
5	0.4644 (20)	0.7024 (20)	0.9918 (8)	2.6164 (8)	2.4792(14)
6	0.5783 (14)	0.8821(12)	0.9550 (16)	2.5195 (16)	2.4775(15)
7	0.5316 (18)	0.8625 (15)	0.9801 (11)	2.5857 (11)	2.5128(11)
8	0.6092 (10)	0.8923 (10)	0.9805 (10)	2.5863 (10)	2.5113(12)
9	0.6944 (2)	0.9348 (6)	0.9938 (6)	2.6221 (6)	2.5851(4)
10	0.6522 (4)	0.8381 (17)	0.9097 (19)	2.4004 (19)	2.3668(19)
11	0.6212 (7)	0.9422 (4)	0.9991(3)	2.6357 (3)	2.5870(3)
12	0.6166 (6)	0.8843 (11)	0.9724 (12)	2.5657 (12)	2.5186(10)
13	0.6756 (3)	0.9431 (3)	0.9901 (9)	2.6113 (9)	2.5666(6)
14	0.5358 (17)	0.7604 (19)	0.8139 (20)	2.1473 (20)	2.1123(20)
15	0.5961 (12)	0.9191 (7)	0.9720 (13)	2.5641 (13)	2.5192(9)
16	0.6297 (6)	0.9372 (5)	0.9951 (5)	2.6253 (5)	2.5822(5)
17	0.5559 (16)	0.8818 (13)	0.9601 (15)	2.5329(15)	2.4756(16)
18	0.5788 (13)	0.8994 (9)	0.9621 (14)	2.5380(14)	2.4878(13)
19	0.5212 (19)	0.8416 (16)	0.9270 (18)	2.4455 (18)	2.3850(18)
20	0.6389 (5)	0.9608 (2)	1.0000 (1)	2.6371 (2)	2.5938(2)

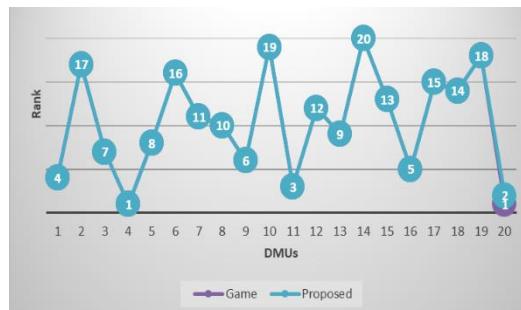


Figure 4. The ranking comparisons for the twenty Thai provinces

4. CONCLUSION

Although the traditional game cross-efficiency method is a valuable approach for tackling DEA ranking challenges, it has limitations in effectively distinguishing all DMUs in specific DEA ranking problems. To address this constraint, we have developed an innovative Gibbs entropy linear programming model that incorporates both optimistic and pessimistic perspectives of the original game cross-efficiency technique. This advancement enables the ranking of all DMUs. Furthermore, the model proficiently quantifies uncertainty when dealing with interval data during DMU ranking. Through the examination of three numerical examples involving six nursing home, numerical example 2, and twenty Thai provinces, we have demonstrated the efficacy of the proposed method in accurately ranking all DMUs. Moreover, the proposed method exhibits a strong correlation with the classical game cross-efficiency method and introduces a novel approach for DEA ranking problems by integrating Gibbs entropy with the traditional game cross-efficiency method. The obtained results, with Spearman's correlation coefficient (r_s) values of 0.998, 0.998, and 0.986 for the proposed method and the classical game cross-efficiency method in the cases of six nursing homes, numerical example 2, and the application involving twenty Thai provinces, respectively, provide compelling evidence of the efficiency and reliability of the proposed method. It offers a more comprehensive rating procedure that leads to sensible and practical conclusions compared to earlier research. Additionally, the proposed Gibbs entropy linear programming model brings added benefits in terms of simplicity and convenience when compared to the original Gibbs entropy optimization model (Lu and Liu's model). Furthermore, we anticipate that our proposed Gibbs entropy linear programming model, developed in this study, can be effectively employed to handle imprecise data. It holds the potential to address problems related to interval data envelopment analysis, fuzzy data envelopment analysis, and multi-criteria decision-making. In summary, the proposed method contributes significantly to DEA ranking methodologies, providing a robust and adaptable framework for decision-making across various domains where uncertainty and imprecise data are prevalent. It offers a more comprehensive and practical approach to ranking, ultimately yielding more dependable and meaningful outcomes compared to prior research.

ACKNOWLEDGEMENTS

The authors extend their sincere appreciation to Kalasin University and Rajamangala University of Technology Isan Khon Kaen Campus for their generous support of this research. Furthermore, the authors express their gratitude to the reviewers for their invaluable feedback and suggestions, which have significantly contributed to enhancing the quality of this paper.

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